DELTA

MATHEMATICAL COLLECTION

The Editor: Mr. Arkady Alt

Thanks to: Mr. Shaul Feder, Mr. Israel Zinger,
Mrs. Tamara Ogen
for their valuable remarks

Translation and Printing: Mr. Maksim Alt

Art: used graphic works of
M. Escher, A. Fomenko, O. Potechina

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The Mathematical Collection "DELTA" is intended for the serious student who is interested in extending his knowledge of mathematical problems and techniques of solutions. It covers a very wide range of mathematical subjects and includes many novel and non-standard approaches.

I recommend it very strongly to the student who wishes to deepen his mathematical understanding, and in particular to those preparing for olympiads and other competitions.

Prof. Joseph Gillis
Weizmann Institute of Science,
Rehovot.

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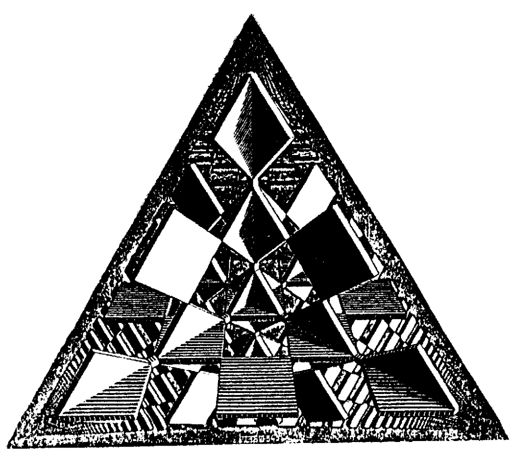
PREFACE

Dear reader !

So, before you another attempt of special mathematical issue for pupils in Israel. In spite of unsuccessful experience of big number of predecessors we hope that this attempt won't fail. We decided to bring this enterprise to a logical end and the mathematical collection of articles of the one author call mathematical magazine for pupils and publish it with not large number of circulation. Absence the word "magazine" on the title is not accidently. All unsuccessful attempts show that maybe just such shape of issue is not appropriate for Israel. We won't analyse the causes, however, the analys has done and taken account within publishing this issue. This first issue can become a first issue in the series "Delta" of mathematical issues for pupils who are interesting in mathematics. Suggested that following issues will be a continuation of preceding issues and despite of structure variety and contents they'll constantly contain some "obligate" departments such as "Delta"'s problems", "Delta"'s competition", "Solutions", "Olympiads", "Delta"'s school". Suggested to include the department on computer science. Suggested presence of department "Information" where can be carried out the reverse connection with the readers.

Our aim is to give a chance to the interested in mathematics and its applications highschool pupils to get a deeper knowledge, extend mathematical outlook, amplify technics for solution of unordinary and complicate problems, accustom to the research, creative work and to the process of cognition. Restriction such pupil by school program and text-book even very good means to stop his development, because just independent, overtime work over things what has been chosen by own desires (not by compulsion), a chance to choose a direction and move on are able to create from an interest the men who's able to create. Of course, here's the greate role plays a teacher who is able to notice such pupil, doesn't miss him and direct him. Therefore, we hope that "mathematical collection" will be useful for teachers too.

Despite we predict most propably objections and wishes to form, contents, design and other attributes of similar issues, we open for critics and we hope the first step will be assess at its true value and on a miracle which could allow the "Mathematical collection" to live for who in the close future become (or not necome) an intellectual power of this state.
Remark.(see "Delta"'s announce" in "Information").



This magazine is not for fun reading which is accessible for who has special alloy of character, interest, diligence and abilities. It allows within a lot of time, pen in the hand and sheet of paper, to be immersed into the wonderful world of harmony and triumph of mind called mathematics.
You reader, no once you feel weak facing difficult problem or place. But don´t despair ! Think indefatigly, try again, look for ways out from deadlock, overcome the top and the light of an idea bright your mind up, and the flash of truth light the dark up and dispel the fog, and a brief instant of happyness reward you and arm you by faith to own power before the face of a new problem, new trial which you wish to yourself. And so, go ahead !

"DELTA" AT THE SCHOOL

In spite of this issue was intended for pupils, i.e. for school, the department "Delta" at the school" suggested to be useful since its has sufficiently special destination - throw light upon questions which are linked with base course of school mathematics, with complicate and delicate places into its, with preparation on graduate exams on mathematics.
We suggest to carry it out by articles, solutions in detail of problems which was offered on graduate exams or similarly (by level) problems. As well in the each issue it suggested to be a "Delta"'s problem s" in which we are going to offer you unordinary problems grouped corresponding to contents of base school courses on mathematics for **9,10,11,12** forms. Solutions of problems of the current issue number will be shown in the next issue.

ORTHOGONAL ELEMENTS IN TRIANGLE

Alt Arkady
"Blikh" school
Ramat-Gan

We shall start from simple and most probably well-known facts.

Theorem 1. The midperpendiculars or perpendiculars drawn to the midpoints of triangle´s sides intersect at one point which is also a centre of the circle circumscribed about triangle.

Theorem 2. The altitudes in triangle intersect at one point which is to be said orthocentre of triangle.

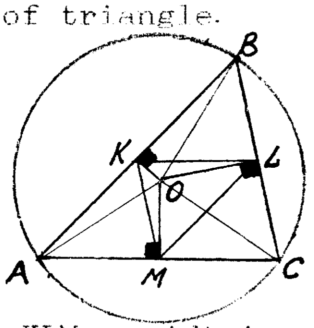
Note 1. If triangle ABC is obtuse then the altitudes and midperpendiculars intersect outside of triangle.

So, we have following objects:
1. Midperpendiculars, points of their intersection which is a centre of the circle, triangle derived by mutually connection midpoints of triangle and naturally call by midtriangle (see *fig.1*).
2. The altitudes h_a, h_b, h_c dropped to sides a,b,c respectively, the point of their intersection is the orthocentre, triangle derived by mutually connection of altitude´s bases is the orthotriangle (see *fig.2*).

Before going on we remind you well-known fact of existence a point into triangle which is equalremoted from triangle´s sides (the point of bisectors intersection) and it follows the center of the circle inscribed into triangle (see *fig.3*).

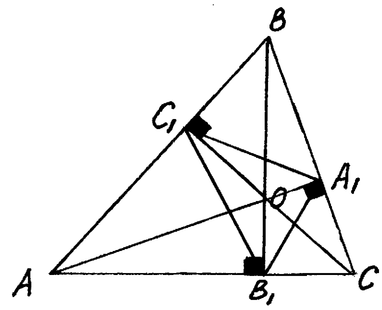
In what follows we shall denote the radius of the circle circumscribed about triangle by R, the radius of the circle inscribed into triangle by r.

As usual S is notion of area of the triangle and $p=\frac{a+b+c}{2}$ is an semiperimeter of triangle.



KLM – *midtriangle*

fig.1



$A_1B_1C_1$ – *orthotriangle*

fig.2

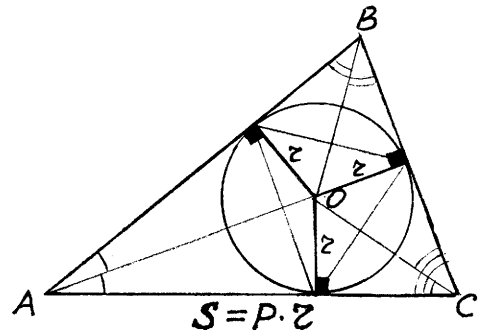


fig.3

Now finished the introduction into region of necessary concepts we shall appeal to the series of problem where the main role play altitudes of triangle and orthogonal elements of triangle, i.e. perpendiculars drawn from some point to the triangle's sides.

We start from geometric correlations (equalities and inequalities) in which altitudes participate.

1. Since $S=\frac{a\cdot h_a}{2}=\frac{b\cdot h_b}{2}=\frac{c\cdot h_c}{2}$, then $a=\frac{2\cdot S}{h_a}$, $b=\frac{2\cdot S}{h_b}$, $c=\frac{2\cdot S}{h_c}$. Hence, in particular

$$a : b : c = \frac{1}{h_a} : \frac{1}{h_b} : \frac{1}{h_c} \qquad (1)$$

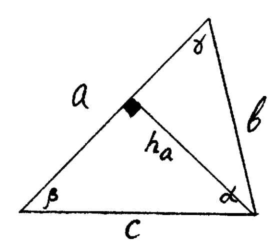
By using this fact we can carry triangle building out by three given ~~sides~~ altitudes.

2. From other hand, $a+b+c = 2p = 2S\cdot\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) \Leftrightarrow \frac{p}{S} = \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} \Leftrightarrow$

$$\Leftrightarrow \qquad \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} \qquad (2)$$

3. $S=\frac{a\cdot b\cdot \sin\gamma}{2} = \frac{2S}{h_a}\cdot\frac{2S}{h_b}\cdot\frac{\sin\gamma}{2} \Leftrightarrow$

$$\Leftrightarrow \qquad S=\frac{h_a\cdot h_b}{2\sin\gamma} \qquad (3)$$



where γ is an angle between the sides a & b (*fig.4*)

fig. 4

4. $h_a = b\cdot\sin\gamma$ and since by the sine theorem $b=2R\cdot\sin\beta$ & $\sin\gamma=\frac{c}{2R}$ we get two formulas for h_a: $h_a=2R\cdot\sin\beta\cdot\sin\gamma$ & $h_a=\frac{b\cdot c}{2R}$.

5. Since by Heron's formula $S=\sqrt{p\cdot(p-a)\cdot(p-b)\cdot(p-c)}$, then

$$h_a=\frac{2S}{a}=\frac{2\cdot\sqrt{p\cdot(p-a)\cdot(p-b)\cdot(p-c)}}{a} \qquad (4)$$

Prove that $h_a \leqslant \sqrt{p\cdot(p-a)}$. (When equality holds?)

6. Prove that if $\frac{1}{h_c}=\frac{1}{a}+\frac{1}{b}$, then $\gamma\le 120^o$.

Solution. Since $h_c=\frac{ab}{2R}$, we get $\frac{2R}{ab}=\frac{1}{a}+\frac{1}{b}$ $\Leftrightarrow$ $a+b = 2R$. Then by cosine theorem $c^2=a^2+b^2-2ab\cdot\cos\gamma$ and by the sine theorem $c=2R\cdot\sin\gamma=(a+b)\cdot\sin\gamma$.
Hence, $(a+b)^2-(a+b)^2\cdot\cos^2\gamma = a^2+b^2-2ab\cdot\cos\gamma \Leftrightarrow 2ab\cdot(1+\cos\gamma)=(a+b)^2\cdot\cos^2\gamma$

$\Rightarrow 2ab\cdot(1+\cos\gamma) \ge 4ab\cdot\cos^2\gamma \Leftrightarrow 1+\cos\gamma \ge 2\cos^2\gamma \Leftrightarrow -\frac{1}{2}\le\cos\gamma\le 1 \Leftrightarrow$

$\Leftrightarrow \gamma \le 120^o$, т.к. $0<\gamma<\pi$.

7. Prove inequality $h_a + h_b + h_c \ge 9\cdot r$.

Solution.

$h_a+h_b+h_c = 2S\cdot(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\cdot\frac{2p}{2p} = \frac{S}{p}\cdot(a+b+c)\cdot(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \ge 9\cdot r$, since

$\frac{S}{p}=r$ and $(a+b+c)\cdot(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \ge 9 \Leftrightarrow \frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}+3 \ge 9 \Leftrightarrow$

$\Leftrightarrow (\frac{a}{b}+\frac{b}{a})+(\frac{a}{c}+\frac{c}{a})+(\frac{b}{c}+\frac{c}{b}) \ge 6$. As long as $\frac{a}{b}+\frac{b}{a}\ge 2$, since

$a^2-2ab+b^2\ge 0 \Leftrightarrow (a-b)^2\ge 0$, then similarly $\frac{a}{c}+\frac{c}{a}\ge 2$ & $\frac{b}{c}+\frac{c}{b}\ge 2$. ■

Notice that each of altitudes in triangle is greater than $2r$. Actually,

$h_a=\frac{2S}{a}$, but $a<p$ since $a<\frac{a+b+c}{2} \Leftrightarrow b+c-a>0$. Thus, $h_a>\frac{2S}{p}=2r$.

8. Prove inequality $\frac{1}{2r}<\frac{1}{h_1}+\frac{1}{h_2}<\frac{1}{r}$, h_1, h_2 are any altitudes of triangle.

9. Prove that if $a<b$, then $a + h_a \le b + h_b$. When does equality occur ?

Solution.

$$\begin{cases} h_a = b\cdot\sin\gamma \\ h_b = a\cdot\sin\gamma \end{cases} \Rightarrow h_a-h_b = (b-a)\cdot\sin\gamma \le b-a, \text{ since } 0<\sin\gamma\leqslant 1.$$

Therefore equality is possible when $\gamma=90^o$, i.e. when triangle is right with a,b its legs and in this case $h_a=b$ & $h_b=a$. (Let try do the same without trigonometry)

10. Let $\triangle ABC$ is triangle with aqute angles and AA_1 & BB_1 be an altitudes drawn from the vertex A & B onto sides BC & AC

respectively (*fig. 5*). $\angle AOB = \angle B_1OA_1 = 180^\circ - \hat{C} = \hat{A} + \hat{B}$, since inside of quadrangle CB_1OA_1 two of the angles are right. And $\angle AOC_1 = \hat{B}$, $\angle BOC = \hat{A}$ as an angles with mutually perpendicular sides. Therefore, if we circumscribe a circle round the triangle AOB with R' is its radii, then by sine theorem we obtain

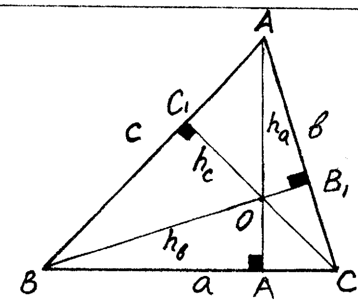


fig. 5

$$2R' = \frac{c}{\sin(\hat{A}+\hat{B})} = \frac{c}{\sin \hat{C}} = 2R,$$ i.e. $R = R'$, where R – radii of the circle circumscribed about triangle ABC. And so,

The circles circumscribed about triangles AOB, BOC, COA *(where* O *is orthocenter) and circle circumscribed about triangle* ABC *have the equal radii.*

Moreover, we derived the values of angles $\angle AOB = \hat{A}+\hat{B}$, $\angle AOC = \hat{A}+\hat{C}$, $\angle COB = \hat{C}+\hat{B}$. We'll see how things will go on in the case of obtuse triangle $\Delta ABC, \angle C > 90^\circ$ (see *fig. 6*).

$\angle OBA = \angle B_1BA = 90^\circ - \hat{A}$, $\angle OAB = \angle A_1AB = 90^\circ - \hat{B} \Rightarrow$ $\angle AOB = 180^\circ - (\angle OAB + \angle OBA) = \hat{A} + \hat{B}$. The result is the same. Compute $\angle AOC$. $\angle AOC = \angle A_1OC = 90^\circ - \angle OCA_1$. But $\angle OCA_1 = \angle BCC_1 = 90^\circ - \angle B$. Hence, $\angle AOC = \angle B$. (We could do it briefly $\angle AOC_1 = \angle A_1BA = \angle B$ as the angles with mutually perpendicular sides. Similarly, $\angle BOC = \angle BOC_1 = \angle A$.

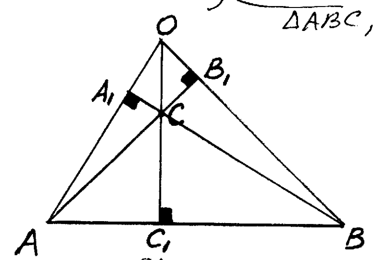


fig. 6

11. We are going to *determine the distance from the vertex of triangle up to the ortocentre* O. To be precise let AO this distance.

Case 1. ABC is an acute triangle (*fig. 7*)

By the sine theorem from triangle AOB

we have $$\frac{AO}{\sin \angle ABO} = 2R \Leftrightarrow \frac{AO}{\sin(90^\circ - \hat{A})} = 2R \Leftrightarrow \frac{AO}{\cos \hat{A}} = 2R \Leftrightarrow AO = 2R \cdot \cos \hat{A} = \frac{a}{\sin \hat{A}} \cdot \cos \hat{A} = a \cdot \text{ctg } \hat{A}.$$ Also

$CO = c \cdot \text{ctg } \hat{C} = 2R \cdot \cos \hat{C}$, $BO = b \cdot \text{ctg } \hat{B} = 2R \cdot \cos \hat{B}$.

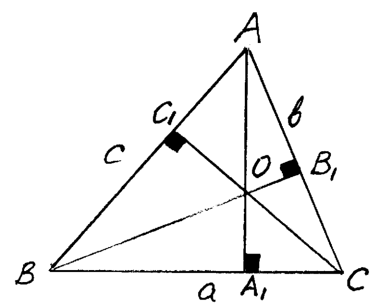


fig. 7

Case 2. ABC is an obtuse triangle. Special situation appear by computing the distance from the orthocentre to vertex of obtuse angle. Consider triangle OAC (see *fig. 8*).

then $\frac{OC}{\sin\angle OAC}=2R \Leftrightarrow \frac{OC}{\sin(90^0-\angle AOA_1)}=2R$

$\Leftrightarrow \frac{OC}{\cos\angle AOB}=2R \Leftrightarrow OC=2R\cdot\cos(\hat{A}+\hat{B})=$

$=-2R\cdot\cos\hat{C}=-\frac{c\cdot\cos\hat{C}}{\sin\hat{C}}=-c\cdot ctg\,\hat{C}.$

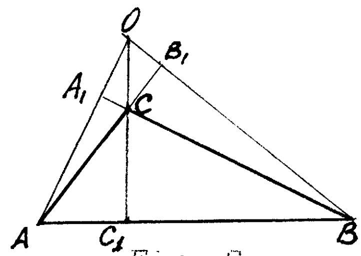


Fig. 8

Hence by summation the results derived in the cases of acute and obtuse triangles:

$$OA = a\cdot|ctg\,\hat{A}| = 2R\cdot|\cos\hat{A}|$$
$$OB = b\cdot|ctg\,\hat{B}| = 2R\cdot|\cos\hat{B}| \qquad (5)$$
$$OC = c\cdot|ctg\,\hat{C}| = 2R\cdot|\cos\hat{C}|$$

12. *Compute the distance from the orthocentre to the sides.* We consider two cases for acute and obtuse triangles.

Case 1. (see *fig. 7*). (ΔABC with aqute angles)

$$OA_1 = h_a-2R\cdot\cos\hat{A} = 2R\cdot\sin\hat{B}\cdot\sin\hat{C} - 2R\cdot\cos\hat{A} =$$
$$= 2R\cdot(\sin\hat{B}\cdot\sin\hat{C} + \cos(\hat{B}+\hat{C})) = 2R\cdot\cos\hat{B}\cdot\cos\hat{C}.$$

Case 2. (*fig. 8*) (ΔABC with one obtuce angle)

$$OC_1 = h_c+OC = h_c-2R\cdot\cos\hat{C} = 2R\cdot\cos\hat{A}\cdot\cos\hat{B};$$
$$OA_1 = OA-AA_1 = 2R\cdot\cos\hat{A} - h_a = -2R\cdot\cos\hat{B}\cdot\cos\hat{C};$$
$$OB_1 = -2R\cdot\cos\hat{B}\cdot\cos\hat{A}$$

Hence we have:

$$\frac{OC}{OC_1}=\left|\frac{\cos\hat{C}}{\cos\hat{A}\cdot\cos\hat{B}}\right|;\quad \frac{OB}{OB_1}=\left|\frac{\cos\hat{B}}{\cos\hat{A}\cdot\cos\hat{C}}\right|;\quad \frac{OA}{OA_1}=\left|\frac{\cos\hat{A}}{\cos\hat{B}\cdot\cos\hat{C}}\right| \qquad (6)$$

(the case of right triangle stroke off as trivial).

Now we can formulate derived results as

a) *The distance from the vertex to the orthcentre of triangle is equal to product of the length of opposite side to the absolute value of cotangent of the angle by this vertex or the product of diameter of the circle circumscribed about triangle to the absolute value of cosine of the angle by this vertex;*

b) *The distance from the ortocentre of triangle to the side of triangle is equal to the product of diameter of the circle circumscribed about triangle to the absolute value of the product of cosines of the angles lying by this side;*

c) *The ratio of the distance from the ortocentre of triangle to a vertex to the distance from the orthocentre to the opposite side is equal to the ratio of cosine of the angle by this vertex to the product of cosines of the angles lying by this side.*

Moreover, we have:

$$AO\cdot OA_1 = BO\cdot OB_1 = CO\cdot OC_1 = 4R^2\cdot |\cos \hat{A}\cdot \cos \hat{B}\cdot \cos \hat{C}| \qquad (7)$$

13. Now the point of us becomes the *orthotriangle*, i.e. triangle in which vertices are the feet of altitudes of the given triangle.
We consider case of acute triangle and leave the case of obtuse triangle for the reader. So, let ABC be an acute triangle, AA_1, BB_1, CC_1 – altitudes, $A_1B_1C_1$ – orthotriangle:

13.1. Since $\angle AC_1C = \angle AA_1C = 90^o$, then points C_1 & A_1 are lying on the circle with diameter AC (see *fig.9*). But then $\angle C_1CA = \angle C_1A_1A = 90^o - \hat{A}$ as inscribed and as leaning on the same arc.
Using similar way for the quardangle ABA_1B_1 and angles $\angle AA_1B_1$ & $\angle ABB_1$ yields equality $\angle AA_1B_1 = \angle ABB_1 = 90^o - \hat{A}$. Similarly,

$\angle A_1C_1C = \angle A_1AC = 90^o - \hat{C}$ & $\angle C_1B_1B = \angle C_1CB = 90^o - \hat{B}$.

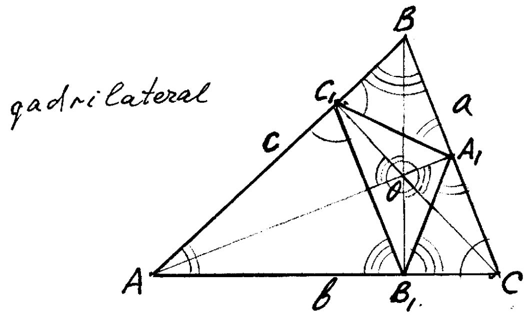


Fig. 9

From this immediately follows that:

a) The altitudes of triangle ABC are bisectors in orthotriangle;
b) Orthocentre O of triangle ABC is the centre of the circle inscribed into orthotriangle. (By the way, what is radius equal to ?)
c) $\angle BC_1A_1 = \angle AC_1B_1 = \hat{C}$; $\angle BA_1C_1 = \angle CA_1B_1 = \hat{A}$; $\angle C_1B_1A = \angle A_1B_1C = \hat{B}$. *It follows that triangles* A_1BC_1, A_1B_1C *and* AB_1C_1 *are similar.*

Compute the ratio of similitude. For this sufficiently to consider one pair of triangles ABC & A_1BC_1. $BC_1 = a\cdot \cos \hat{B}$, $BC=a$. Hence, the ratio of similitude of the triangles ABC & A_1BC_1 is equal to $\cos \hat{B}$. Similarly, $A_1B_1C \simeq ABC$ with the ratio $\cos \hat{C}$ and $ABC \simeq AB_1C_1$ with the ratio $\cos \hat{A}$. Thus,

$$A_1C_1 = b\cdot \cos \hat{B}, \quad A_1B_1 = c\cdot \cos \hat{C}, \quad B_1C_1 = a\cdot \cos \hat{A}.$$

Compute the area of the orthotriangle $A_1B_1C_1$. If S is the area of triangle ABC, then

$$S_{A_1B_1C_1} = S - S\cdot(\cos^2\hat{A} + \cos^2\hat{B} + \cos^2\hat{C}) = S\cdot(1 - \cos^2\hat{A} - \cos^2\hat{B} - \cos^2\hat{C}) =$$
$$= 2S\cos\hat{A}\cdot\cos\hat{B}\cdot\cos\hat{C}$$

Compute the semiperimeter of the orthotriangle:

$$p = \frac{a\cdot \cos \hat{A} + b\cdot \cos \hat{B} + c\cdot \cos \hat{C}}{2}.$$

From this we can compute the radius of the circle inscribed into orthotriangle

13.2. But the orthotriangle possesses one more wonderful exclusive property
The perimeter of the orthotriangle is the smallest perimeter among all perimeters of triangles inscribed into given triangle. (Triangle KLM is to be

said inscribed into given triangle if on each side of given triangle lie exactly one vertex of the triangle KLF).
This statement known as Faniano's theorem. We are going to prove this theorem. For any inscribed triangle KLM into triangle ABC we shall denote the perimeter of this triangle by P(K,L,M). Situate the points: point K lies on AB, L lies on AC and M lies on BC. Then the theorem can be rewritten briefly

Find points E, H, F *on the sides* AB, AC, BC *respectively, such that:*

$$P(E,H,F) = \min_{\substack{K\in AB\\ L\in AC\\ M\in BC}} P(K,L,M)$$

But since $\min_{K,L,M} P(K,L,M) = \min_{L} \{\min_{K,M} P(K,L,M)\}$ we'll prove it by following way. Fix the point L on the side AC and find $\min_{K,M} P(K,L,M)$. The minimum, of course, will depend from L as depend from L the location of points K & M on the sides AB & CD, i.e. we get some function $d(L)=\min_{K,M} P(K,L,M)$ and location of points K(L) & M(L) (this notion shows dependence of location K & M from L). Than we find location of point H on AC such that $d(H)=\min_{L\in AC} d(L)$ and in this case E=K(H) & F=M(H). Actually, $d(H)\le d(L)\le P(K,L,M)$. But by definition d(H) is the perimeter of the triangle EFH. This reasoning is the ground of further actions and also is the plan of solution of this problem.

Let the point L be situated by arbitrary location on the side AC (see *fig.10*), K & M be taken arbitrary on the sides AB & BC.
Let L_1 & L_2 be the symmetric points to the L relatively lines AB & BC. LL_1 is perpendicular to AB, intersect AB at the point S and bisecting itself by S; LL_2 is perpendicular to BC, intersect BC at the point T and bisecting itself by T. Connect the points L_1 & L_2. This segment intersect the sides AB and CB at the points K_1 и M_1 respectively. Connect them with L. Now look what we got:

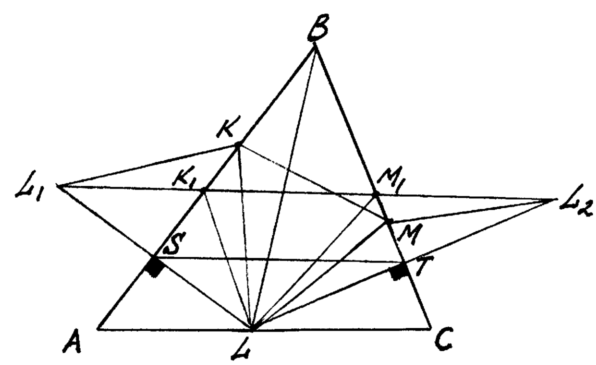


Fig. 10

So, $KL_1=KL$ & $ML_2=ML$, therefore $KL_1+KM+ML_2 = P(K,L,M)$. But $L_1L_2\le KL_1+KM+ML_2$ as line segment connecting broken line's vertices. Moreover, location of points L_1, L_2, K_1 & M_1 unique depends from location of L on AC only (it does not depend from location of K and M).
$L_1K_1+K_1M_1+M_1L_2 = LK_1+K_1M_1+M_1L = P(K_1,L,M_1)$. Thus, independently from the location K & M on the sides AB & AC: $P(K_1,L,M_1)\le P(K,L,M)$. In other words,

$P(K_1,L,M_1) = \min_{K,M} P(K,L,M)$.

But $ST=\frac{1}{2}\cdot L_1L_1$ and it is a diagonal in quadrangle LSBT which can be inscribed into circle so that the sum of opposite angles would be equal to 180^o, with BL is diameter of this circle. By the sine theorem $ST=BL\cdot\sin\hat{B}$, i.e. $d(L)=L_1L_2=2\cdot ST=2\cdot BL\cdot\sin\hat{B}$ depend exclusively from the length of BL and get the smallest value d(H) only when L is the feet H of the altitude drawn from B to the side AC. (fig. 11)

We show that if L is the feet H of the altitude, then E=K(H) & F=M(H) will be the feet of the altitudes drawn to the sides AB & BC respectively i.e. we're going to prove that EFH is the orthotriangle (*fig. 11*).

As said above, the quadrangle HSBT inscribed into circle. Connect S with T then, $\angle SHB=\angle STB=\hat{A}$. Similarly, $\angle TSB=\hat{C}$. But since H_1H_2 is parallel to ST (ST is the midline in the triangle H_1HH_2)

then $\hat{A}=\angle BFE=\angle CFH_2=\angle HFT$ & $\hat{C}=\angle BEF=\angle H_1ES=\angle SEH$.

Thus, $\angle HEF=180^o-2\cdot\hat{C}$, $\angle EFH=180^o-2\cdot\hat{A}$ and it means that $\angle EHF=180^o-2\cdot\hat{B}$, $\angle EHA=\angle FHC=\hat{B}$.

So, triangles AEH, FCH, FEB are mutually similar and each one of them is similar to the triangle ABC, with $AH=c\cdot\cos\hat{A}$. (because BH is altitude)

Thus, $AE=b\cdot\cos\hat{A}$ and it means that E is the feet of altitude dropped from C to AB.

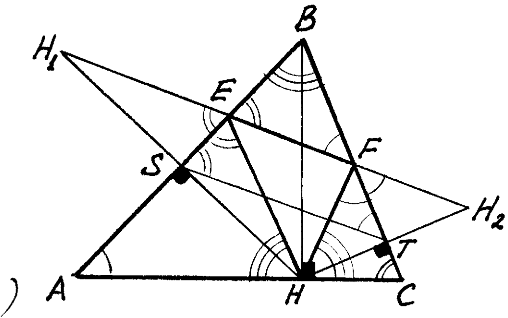


Fig. 11

Similarly, $CH=a\cdot\cos\hat{C}$ & $CF=b\cdot\cos\hat{C}$ and it means that F is the feet of altitude dropped from A to the side BC. Here we used that a side of triangle and its projection determine the altitude (such determination is unique).

I think that we've talked enough about orthotriangle, we arrived to the third part of this article which is linked with properties of the distance from arbitrary point in triangle to the its sides.

Denote by d_a, d_b, d_c the distances from arbitrary point O inside triangle ABC to the its sides respectively, and consider following problems with those values.

1. Prove $\min\{h_a,h_b,h_c\}\le d_a+d_b+d_c\le\max\{h_a,h_b,h_c\}$.

When does equality occur ? (How do things go for equilateral triangle ?)

2. Find such point O inside given triangle so that the sum of squares of the distances from this point to the sides of triangle will be greatest.

3. Prove

$$d_a\cdot d_b\cdot d_c\le\frac{8\cdot S^3}{27\cdot abc}$$

When does inequality turn to equality ?

For the solving all those problem will be useful the formula (also this formula is useful in other situations).

$$S = \frac{a\cdot d_a}{2} + \frac{b\cdot d_b}{2} + \frac{c\cdot d_c}{2} = \frac{a\cdot d_a + b\cdot d_b + c\cdot d_c}{2}.$$

Solutions. (see *fig.12*)

Problem 1. Let $a=\min\{a,b,c\}$, then $h_a = \frac{2S}{a} = \max\{h_a,h_b,h_c\}$ and

$$S \ge \frac{a}{2}\cdot(d_a+d_b+d_c) \iff \frac{2S}{a} = h_a \ge d_a+d_b+d_c$$

Let $a=\max\{a,b,c\}$, then $h_a = \frac{2S}{a} = \min\{h_a,h_b,h_c\}$ and

$$S \le \frac{a}{2}\cdot(d_a+d_b+d_c) \iff \frac{2S}{a} \le d_a+d_b+d_c \iff \min\{h_a,h_b,h_c\} \le d_a+d_b+d_c.$$

So, $\min\{h_a,h_b,h_c\} \le d_a+d_b+d_c$.

We're interesting in cases of equality.

a) The triangle is equilateral, then $\min\{h_a,h_b,h_c\}=\max\{h_a,h_b,h_c\}$, and it follows that for any point O inside triangle $d_a+d_b+d_c=h$, where h is altitude of equilateral triangle.

b) Let $a<b=c$, and right inequality take place then $h_a=\max\{h_a,h_b,h_c\}$, $d_a+d_b+d_c=h_a$. If O coincides with the vertex A, then $d_b=d_c=0$ & $d_a=h_a$.

Let O be not coincided with the vertex A, then d_b or $d_c\ne 0$, $\frac{a}{2}\cdot(d_a+d_b+d_c)=S$

From other hand $S=\frac{a}{2}\cdot d_a+\frac{b}{2}\cdot d_b+\frac{c}{2}\cdot d_c$. Hence,

$$\frac{d_b\cdot b+d_c\cdot c+d_a\cdot a}{2} = S > \frac{d_a\cdot a + d_b\cdot a + d_c\cdot a}{2} = S.$$ We got contradiction.

Consider now the right-hand side of inequality. In the case of equality we have $d_a+d_b+d_c=h_b=h_c$ $(b=c)$. Let O be lying on the side a. Then $d_a=0$ and $d_b+d_c=h_b=h_c$ (prove it by yourself !).

Suppose that the point O isn't lying on the side a, then $d_a>0$

In this case:

$$\frac{b\cdot(d_b+d_c) + a\cdot d_a}{2} = S < \frac{b}{2}\cdot(d_a+d_b+d_c) = \frac{b}{2}\cdot h_b = S.$$ We got contradiction.

So, if $a<b=c$, then equality $\min\{h_a,h_b,h_c\}=d_a+d_b+d_c$ is possible when point O lies on the side a; equality $\max\{h_a,h_b,h_c\}=d_a+d_b+d_c$ is possible if and only if the point O coincides with the vertex A.

The case for isosceles triangle $a=b<c$ we leave for the reader.

We consider the case of arbitrary triangle (all sides are distinct).

Set $a<b<c$, then $h_a>h_b>h_c$. Consider case of equality $h_a=d_a+d_b+d_c$

If O does not coincide with A, then either d_a or d_c are not equal to zero.

Then $S = \frac{a \cdot h_a}{2} = \frac{a \cdot (d_a + d_b + d_c)}{2} < \frac{a \cdot d_a + b \cdot d_b + c \cdot d_c}{2} = S$. We got contradiction.

So, equality $h_a = d_a + d_b + d_c$ (handwritten above: $= \max\{h_a, h_b, h_c\} =$) is possible only when point O coincides with A.

In the case of equality $h_c = d_a + d_b + d_c$ we do the same. If O coincides with C then, $d_c = h_c$ & $d_a = d_b = 0$. If point O doesn't coincide with C, then $d_c < h_c$ & d_a or $d_b \neq 0$. Then

$$S = \frac{h_c \cdot c}{2} = \frac{c \cdot (d_c + d_a + d_b)}{2} > \frac{c \cdot d_c + a \cdot d_a + b \cdot d_b}{2}.$$ But again we got contradiction.

And so, equality $h_c = \min\{h_a, h_b, h_c\} = d_a + d_b + d_c$ is possible only when the point O coincides with the vertex C.

Problem **2**. Use Cauchy-Bunyakovsky inequality (see the article in this issue in department "Delta"'s school") to expression $2S = c \cdot d_c + b \cdot d_b + a \cdot d_a$. We have:

$$4S^2 = (c \cdot d_c + b \cdot d_b + a \cdot d_a)^2 \leq (c^2 + b^2 + a^2) \cdot (d_c^2 + d_b^2 + d_a^2)$$

Hence $d_a^2 + d_b^2 + d_c^2 \geq \frac{4 \cdot S^2}{a^2 + b^2 + c^2}$. And equality occur if $\frac{d_a}{a} = \frac{d_b}{b} = \frac{d_c}{c}$. Let k be a factor of proportionality, then $d_a = ka$, $d_b = kb$, $d_c = kc$.

Show that such point exist in triangle.

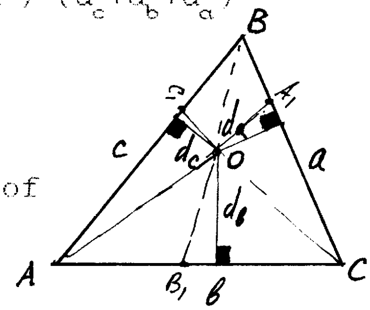


Fig. 12, 13.

Let O be such point that occur those equalities. Draw through the point O from vertices A, B & C lines up to intersection with the sides at points A_1, B_1 & C_1. (fig. 13)

$$S_{AOB} = \frac{c \cdot d_c}{2} = \frac{kc^2}{2}; \quad S_{BOC} = \frac{a \cdot d_a}{2} = \frac{ka^2}{2}; \quad S_{AOC} = \frac{b \cdot d_b}{2} = \frac{kb^2}{2}.$$

Then

$$\frac{S_{AOB}}{S_{BOC}} = \frac{c^2}{a^2} = \frac{AB_1}{B_1C}; \quad \frac{S_{AOB}}{S_{AOC}} = \frac{c^2}{b^2} = \frac{BA_1}{A_1C}$$ and similarly, $$\frac{BC_1}{C_1A} = \frac{a^2}{b^2} = \frac{S_{BOC}}{S_{AOC}}.$$

Now, prove that if we take on the sides of triangle ABC points A_1, B_1 & C_1 such, that: $\frac{AB_1}{B_1C} = \frac{c^2}{a^2}$; $\frac{CA_1}{A_1B} = \frac{b^2}{c^2}$; $\frac{BC_1}{C_1A} = \frac{a^2}{b^2}$, then lines AA_1, BB_1, CC_1 intersect at one point (this point called *Lemuan's point of the triangle* ABC).

Suppose that we already drew lines AA_1 & BB_1 and $\frac{CA_1}{A_1B} = \frac{b^2}{c^2}$ & $\frac{AB_1}{B_1C} = \frac{c^2}{a^2}$.

Let O be the point of intersection of those lines. Draw the line trough point O and C up to intersection with AB at point C_1. Prove that $\frac{AC_1}{C_1B} = \frac{b^2}{a^2}$.

Then, as easily to notice $\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = \frac{S_{AOB}}{S_{BOC}} \cdot \frac{S_{COA}}{S_{AOB}} \cdot \frac{S_{BOC}}{S_{COA}} = 1$.

From other hand: $\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} = \frac{c^2}{a^2} \cdot \frac{b^2}{c^2} = \frac{b^2}{a^2}$. Hence, $\frac{AC_1}{C_1B} = \frac{b^2}{a^2}$.

(Using the idea of this proof the reader can easily prove *Cheve's Theorem:* On sides of triangle ABC taken points: A_1 on BC, C_1 on AB, B_1 on AC. For the segments AA_1, BB_1 & CC_1 to be intersect at one point O it is necessary and sufficient that

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = 1$$

such segments used to call *chevians* and the points of triangle A_1, B_1, C_1 for which AA_1, BB_1, CC_1 are chevians call *concurrents*).

We offer the reader to carry out the constructing of point O in the problem.

And, finally, notice that the smallest value of sum $d_a^2+d_b^2+d_c^2$ is equal to $\frac{4S^2}{a^2+b^2+c^2}$ and reaches this value when $k=\frac{2S}{a^2+b^2+c^2}$ ($\Leftarrow S_{AOB}+S_{BOC}+S_{COA}=S$)

As exercise we offer to the reader series of geometric inequalities and their corollaries are another reachable lower bound for the sum of squares of the distance from arbitrary point of triangle to its sides. However, this lower bound can be reached if we refuse from complete determination of triangle, i.e. for example determination of all its sides.

1. Prove

a) $\sin\frac{\hat{A}}{2}\cdot\sin\frac{\hat{B}}{2}\cdot\sin\frac{\hat{C}}{2} \le \frac{1}{8}$, where $\hat{A}, \hat{B}, \hat{C}$ are angles of triangle with sides a,b,c (*Hint. Prove that* $\sin\frac{\hat{A}}{2} \le \frac{a}{2\sqrt{bc}}$ *by the cosine theorem*).

b) $\cos\frac{\hat{A}}{2}\cdot\cos\frac{\hat{B}}{2}\cdot\sin\frac{\hat{C}}{2} \le \frac{1}{8}$. (*Hint. Set* $\alpha = \frac{\pi}{2}-\hat{A}$, $\beta = \frac{\pi}{2}-\hat{B}$, $\gamma = \frac{\pi}{2}-\hat{C}$)

c) $\sin^2\frac{\hat{A}}{2} + \sin^2\frac{\hat{B}}{2} + \sin^2\frac{\hat{C}}{2} \geq \frac{3}{4}$. Equality is reachable when $\hat{A}=\hat{B}=\hat{C}$.

(*Hint. Prove* $\sin^2\frac{\hat{A}}{2}+\sin^2\frac{\hat{B}}{2}+\sin^2\frac{\hat{C}}{2} = 1-2\sin\frac{\hat{A}}{2}\cdot\sin\frac{\hat{B}}{2}\cdot\sin\frac{\hat{C}}{2}$)

d) $a^2 + b^2 + c^2 \le 9\cdot R^2$. Find the condition of appearance of the equality.

2. Prove: *a)* $p = r\cdot(\mathrm{ctg}\frac{\hat{A}}{2}+\mathrm{ctg}\frac{\hat{B}}{2}+\mathrm{ctg}\frac{\hat{C}}{2})$, p - semiperimeter, r - radius of the circle inscribed into triangle.

б) $S = 2R^2\cdot\sin\hat{A}\cdot\sin\hat{B}\cdot\sin\hat{C}$, S - the area of triangle; R - radius of the circle circumscribed about triangle.

3. If $\hat{A},\hat{B},\hat{C}$ are the angles of triangle, then

$$a)\quad \mathrm{tg}\,\hat{A} + \mathrm{tg}\,\hat{B} + \mathrm{tg}\,\hat{C} = \mathrm{tg}\,\hat{A}\cdot\mathrm{tg}\,\hat{B}\cdot\mathrm{tg}\,\hat{C}$$

$$б)\quad \mathrm{ctg}\frac{\hat{A}}{2} + \mathrm{ctg}\frac{\hat{B}}{2} + \mathrm{ctg}\frac{\hat{C}}{2} = \mathrm{ctg}\frac{\hat{A}}{2}\cdot\mathrm{ctg}\frac{\hat{B}}{2}\cdot\mathrm{ctg}\frac{\hat{C}}{2}$$

4. Prove:

$$r = 4R\cdot\sin\frac{\hat{A}}{2}\cdot\sin\frac{\hat{B}}{2}\cdot\sin\frac{\hat{C}}{2}$$

5. $R \geq 2\cdot r$, when equality occur only in the case of equilateral triangle. (Try to find a geometric proof).

6. Prove $S \geq 3\sqrt{3}\cdot r^2$. (*Hint. Use Heron's formula for* S; *use equality* $S=p\cdot r$ *and Cauchy's inequality* (see article "Variation on inequality theme")).

7. Prove

$$\mathrm{tg}\,\hat{A}\cdot\mathrm{tg}\,\hat{B}\cdot\mathrm{tg}\,\hat{C} \geq 3\sqrt{3} \quad \text{и} \quad \mathrm{ctg}\frac{\hat{A}}{2}\cdot\mathrm{ctg}\frac{\hat{B}}{2}\cdot\mathrm{ctg}\frac{\hat{C}}{2} \geq 3\sqrt{3}.$$

Using inequalities $S \geq 3\sqrt{3}\cdot r^2$ & $a^2 + b^2 + c^2 \leq 9\cdot R^2$ we get:

$$d_a^2 + d_b^2 + d_c^2 \geq \frac{4\cdot S^2}{a^2+b^2+c^2} \geq 12\cdot\frac{r^4}{R^2}$$

In order to in this inequality occur equality sufficient that $S=3\sqrt{3}\cdot r^2$ & $a^2+b^2+c^2=9\cdot R^2$, that is possible if and only if triangle is equilateral and in this case equality $d_a^2+d_b^2+d_c^2 = \frac{4\cdot S^2}{a^2+b^2+c^2}$ holds if $d_a:d_b:d_c=a:b:c=1:1:1$, i.e. when the point O coincides with the centre of the circle inscribed into triangle ABC.

Problem 3. By *AM-GM* inequality:

$$\frac{2}{3}\cdot S = \frac{a\cdot d_a+b\cdot d_b+c\cdot d_c}{3} \geq \sqrt[3]{abc\cdot d_a\cdot d_b\cdot d_c} \iff \frac{8}{27}\cdot S^3 \geq abc\cdot d_a d_b d_c \iff$$

$$\iff \frac{8}{27}\cdot\frac{S^3}{abc} \geq d_a\cdot d_b\cdot d_c \iff 27\cdot d_a\cdot d_b\cdot d_c \leq h_a\cdot h_b\cdot h_c.$$ Equality occur when $a\cdot d_a=b\cdot d_b=c\cdot d_c$, i.e. $d_a:d_b:d_c=\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$.

By straightforward division equality $2\cdot S = a\cdot d_a+b\cdot d_b+c\cdot d_c$ by $2\cdot S$ we get

$1 = d_a \cdot \frac{a}{2S} + d_b \cdot \frac{b}{2S} + d_c \cdot \frac{c}{2S}$ or

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = 1$$

As exercises prove follows inequalities and find out when equality occur:

1. $\frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \geq 9.$

2. $(h_a - d_a) \cdot (h_b - d_b) \cdot (h_c - d_c) \geq 8 \cdot d_a \cdot d_b \cdot d_c$

Prove *Carno's theorem*:
In arbitrary triangle the sum of distance from the centre of circle circumscribed about triangle to its sides is equal to the sum of radiuses of the inscribed and circumscribed circles, i.e. $d_a + d_b + d_c = r + R$.

(Hint. Express d_a, d_b, d_c, r *through* R *and angles of triangle*).

Now one more time go back to the most important orthogonal element of triangle - altitude.
As before the solutions of concrete problems are the fundamental way of presentation information.

Problem. Prove $\max_{a+b+c=2p} \min \{h_a, h_b, h_c\} = 3 \cdot r$, where semiperimeter p is given.

Solution.

Let $a \geq b \geq c$, then $h_a \leq h_b \leq h_c$. In supposition $a \geq b \geq c$ left to prove $h_a \leq 3 \cdot r$.

Since $a \cdot h_a = 2 \cdot S = 2r \cdot p$, then $h_a = r \cdot \frac{2p}{a}$. But $2p = a+b+c$ and it means that

$\frac{a+b+c}{a} = 1 + \frac{b}{a} + \frac{c}{a} \leq 1+1+1 = 3$. Equality occur when $b=a$ & $c=a$, i.e. in the case of equilateral triangle.

Problem. Prove inequality:

(fig. 14) $$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq 1 + \frac{R}{r}$$ (there m_a, m_b, m_c are medians of triangle on the sides a, b, c respectively)

Solution. Let O be the centre of circle radii R circumscribed about triangle ABC, d_a, d_b, d_c are distance from the centre to the sides a, b, c respectively.

$AA_1 = m_a$; $OA = R$; $OA_1 = d_a$;

From the triangle inequality $AA_1 \geq OA + OA_1$

or it can be written $m_a \leq R + d_a$

Similarly, $m_b \leq R + d_b$; $m_c \leq R + d_c$.

Hence,

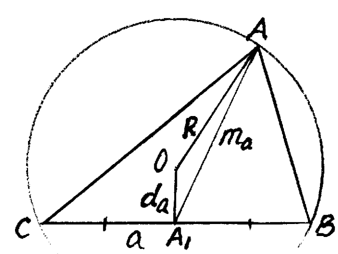


fig 14

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq \frac{R+d_a}{h_a} + \frac{R+d_b}{h_b} + \frac{R+d_c}{h_c} = R \cdot \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) + \frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = \frac{R}{r} + 1$$

Equality occur if triangle inequality turn to equalities, i.e. when O lies on the medians and then median coincide with altitude..

Therefore, only in equilateral triangle this inequality turns to equality. Attentive reader probably paid attention that in proceed of solutions appeared secondary characteristics of triangle like p,r and R. However, those three value are determining the triangle no worse then its sides a,b,c. In particular, we're interesting in connection between altitudes of triangle and values p,r and R. The role of mediator here will play the area S, moreover we need formulas setting connection between length of the sides of triangle a,b,c with p,r and R.
We want to express $h_a+h_b+h_c$, $h_a\cdot h_b+h_a\cdot h_c+h_b\cdot h_c$, $h_a\cdot h_b\cdot h_c$ through p,r & R

We shall start from most simple thing

$$h_a\cdot h_b + h_a\cdot h_c + h_b\cdot h_c = \frac{4S^2}{ab} + \frac{4S^2}{bc} + \frac{4S^2}{ac} = \frac{4S^2}{abc}\cdot(a+b+c) = \frac{4S^2\cdot 2p}{abc}.$$

But $S=p\cdot r$ and from the sine theorem

$$S = \frac{ab\cdot\sin\hat{C}}{2} = \frac{abc\cdot\sin\hat{C}}{2\cdot c} = \frac{abc}{4R}.$$ Hence,

$$h_a\cdot h_b + h_a\cdot h_c + h_b\cdot h_c = \frac{4S}{abc}\cdot pr\cdot 2p = \frac{2p^2r}{R}.$$

Find a formula which can express S through $h_a\cdot h_b\cdot h_c$.

Since $S=\frac{h_a\cdot h_b}{2\sin\gamma}$, then $S^2=\frac{h_a\cdot h_b\cdot S}{2\cdot\sin\gamma} = \frac{h_a\cdot h_b\cdot h_c\cdot c}{4\cdot\sin\gamma} = \frac{h_a\cdot h_b\cdot h_c\cdot 2R}{4} = \frac{1}{2}R\cdot h_a h_b h_c.$

Hence, $S=\sqrt{\frac{1}{2}R\cdot h_a h_b h_c}$ and $h_a\cdot h_b\cdot h_c = \frac{2S^2}{R} = \frac{2r^2p^2}{R}$.

Left to express the sum $h_a+h_b+h_c$ through p,r & R:

$$h_a + h_b + h_c = \frac{bc}{2R} + \frac{ac}{2R} + \frac{ab}{2R} = \frac{1}{2R}\cdot(ab+ac+bc).$$

By Heron's formula $S^2=p\cdot(p-a)\cdot(p-b)\cdot(p-c)$ or $\frac{S}{p}\cdot S = r^2\cdot p=(p-a)\cdot(p-b)\cdot(p-c)$ $= p^3 - p^2\cdot(a+b+c) + p\cdot(ab+ac+bc) - abc$. But $abc = 4\cdot S\cdot R = 4\cdot p\cdot r\cdot R$, $a+b+c=2p$.

It follows that $p\cdot r^2 = -p^3 - 4prR + p\cdot(ab+bc+ac)$. Hence,

$ab+bc+ac = r^2 + p^2 + 4rR$. Therefore, $h_a + h_b + h_c = \frac{r^2+p^2+4rR}{2R}$.

By derived formulas

$$\begin{cases} h_a+h_b+h_c = \frac{r^2+p^2+4rR}{2R} \\ h_a\cdot h_b+h_a\cdot h_c+h_b\cdot h_c = \frac{2p^2r}{R} \\ h_a\cdot h_b\cdot h_c = \frac{2r^2p^2}{R} \end{cases}$$

and by Wiette's theorem for cubic equation we make following very important conclusion:

The altitudes h_a, h_b, h_c *are the roots of cubic equation:*

$$h^3 - \frac{r^2+p^2+4rR}{2R}\cdot h^2 + \frac{2p^2r}{R}\cdot h - \frac{2r^2p^2}{R} = 0$$

it can be rewritten like this

$$2R\cdot h^3 - (r^2+p^2+4rR)\cdot h^2 + 4p^2r\cdot h - 4r^2p^2 = 0$$

By the way we got following statement:
The lengths of sides of the triangle are the roots of cubic equation

$$x^3 - 2p\cdot x^2 + (r^2+p^2+4rR)\cdot x - 4prR = 0$$

Similarly equation we can get for values p-a, p-b, p-c, for the trigonometric function of the same name for angles, for double angles, for half angles of the triangle.

At the end of this brief tour in the country of orthogonal elements in triangle we look at construction which gives the generalization of orthotriangle. Namely, for an arbitrary point O taken on the plane, the points A_1, B_1, C_1 are feet of perpendiculars drawn from the point O to the sides (or to their outside part) BC,AC,AB of the triangle ABC. If those points lie on a line, then they are forming a triangle $A_1B_1C_1$ which is calling *pedal triangle* of the given triangle ABC and point O. Clearly, for the point O which is the point of intersection of altitudes, the pedal triangle is the orthotriangle. In general case, if the points A_1, B_1, C_1 lie on a line then in what follows we shall say that these points are *collinear*. In this we may say that the triangle $A_1B_1C_1$ is a *singular pedal triangle.* Of course, we're interesting in the cases when the pedal triangle is not singular.

The following theorem gives an exhaustive answer on this question:

Theorem. Given triangle ABC and point O taken on the plane. Their pedal triangle $A_1B_1C_1$ is singular if and only if O lies on the circle circumscribed about triangle ABC.

Proof.

Necessity. Let pedal triangle be singular, i.e. A_1, B_1, C_1 as feet of the perpendiculars dropped from the O to the sides BC, AC, AB, are collinear or they are lying on a line. We're going to show that inthis case the point O must lie on the circle circumscribed about triangle ABC. (see *fig.15*).

Connect the point O with B and C.

1. Since OC_1 perpendicular to OB_1 and OB_1 perpendicular to AC, then $\angle C_1OB_1 = 180^o - \hat{A}$

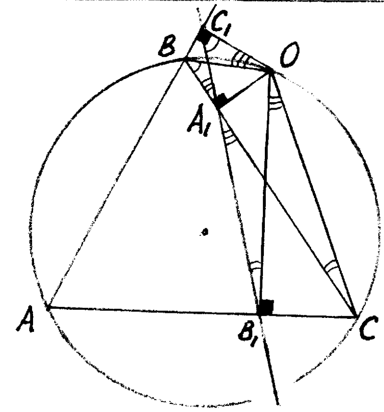


Fig. 15

2. About quadrangle OC_1BA_1 we can circumscribe a circle, then $\angle OC_1A_1=\angle OBA_1$, as inscribed and leaning on the same arc angles.

3. About quadrangle $OA_1B_1C_1$ we can circumscribe a circle with diameter OC, then $\angle A_1CO=\angle A_1B_1O$.

4. From the 1,2,3 follows that in triangles C_1OB_1, BOC the angles $\angle C_1OB=\angle BOC=180^{o}-\hat{A}$

But then about quadrangle BACO we can circumscribe a circle. It follows that the point O lies on the circle circumscribed about triangle ABC.

Sufficiency. Prove inverse theorem. Let point O be lying on the circle circumscribed about triangle ABC, then, $\angle BOC=180^{o}-\hat{A}$. Since OC_1 perpendicular to AB and OB_1 perpendicular to AC, then $\angle C_1OB_1=180^{o}-\hat{A}=\angle BOC$. Repeat parts 2 and 3 of the proof of necessity and we get

$\angle C_1A_1B_1=360^{o}-(\angle OC_1A_1+\angle OB_1A_1+\angle C_1OB_1)=360^{o}-(\angle OBA_1+\angle OCA_1+\angle BOC)=360^{o}-180^{o}=180^{o}$.

So we get that points C_1, A_1, B_1 lie on a line. This line call *Simson's line.* And so, the theorem is proved. ■

Hence, this theorem is to be said the *test of collinearity* of points A_1, B_1, C_1 which are feet of perpendiculars drawn from the point to the sides of triangle ABC.

Let O be an arbitrary point in the plane. We compute the lengths of sides of the pedal triangle (we don't care about singularity), or compute lengths of segments A_1B_1, A_1C_1, B_1C_1. Denote by δ_A, δ_B, δ_C the distances from the point O to the vertices of triangle ABC.

$OA=\delta_A$, $OC_1=d_c$, $OB_1=d_b$ (see *fig. 16*).

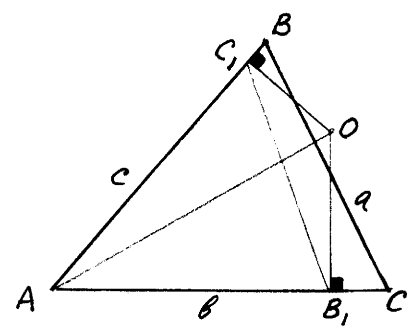


Fig. 16

Since in the quadrangle AC_1OB_1 angles $\angle AC_1O$ & $\angle OB_1A$ are right, then we can circumscribe a circle with diameter OA about this quadrangle. But then by the sine theorem for B_1C_1 which is the side of triangle AC_1B_1 and lying opposite to angle $\hat{A}$ we get

$$\frac{C_1B_1}{\sin\hat{A}} = OA \Rightarrow C_1B_1=\delta_A\cdot\sin\hat{A},\ \hat{A} \text{ we can get from triangle ABC } \frac{a}{\sin\hat{A}}=2R,$$

where R is radii of the circle circumscribed about triangle ABC.

Thus, $C_1B_1=\dfrac{\delta_A\cdot a}{2R}$. Similarly we obtain $A_1B_1=\dfrac{\delta_C\cdot c}{2R}$, $A_1C_1=\dfrac{\delta_B\cdot b}{2R}$.

We fix an arbitrary point O in the triangle ABC, then the pedal triangle $A_1B_1C_1$ is not singular and point O lies inside of that triangle, since $\angle C_1OA_1=180^o-\hat{B}$, $\angle C_1OB_1=180^o-\hat{A}$, $\angle A_1OB_1=180^o-\hat{C}$, and it follows that $\angle C_1OA_1+\angle C_1OB_1+\angle A_1OB_1 = 3\cdot 180^o-180^o = 360^o$.

(Prove the test by yourself. Point O lies inside triangle if and only if every side seen by angle which smaller than 180^o and the sum of those angles is equal to 360^o).

For triangle $A_1B_1C_1$ and point O we create the new pedal triangle $A_2B_2C_2$ (the second pedal triangle) (see *fig. 17*).

As said above, point O lies inside triangle $A_2B_2C_2$. Finally, in the triangle $A_2B_2C_2$ we build the third pedal triangle $A_3B_3C_3$ as well not singular and with the O inside its. As it turn out, the third pedal triangle and given triangle are similar.

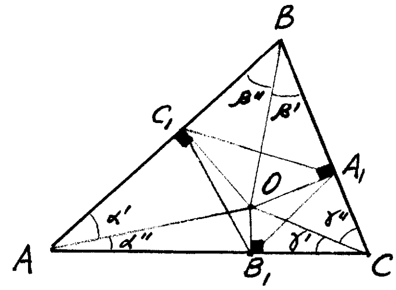


Fig. 17

Described above procedure of constructing the first, second, third and so on pedal triangles is typical iteration. And for further we need to look well at behavior of angles by iterations. Sufficient to consider one step from the triangle ABC to the triangle $A_1B_1C_1$.

Connect point O with vertices A, B & C (*fig. 18*) then $\angle BAO=\angle C_1AO=\angle C_1B_1O$ as long as $\angle C_1AO$ & $\angle C_1B_1O$ are inscribed and leaning on the same arc of the circle circumscribed about quadrangle AC_1OB_1 (since angles $\angle AC_1O$ & $\angle AB_1O$ are right). By the same reason

we get $\angle CAO=\angle BAO=\angle B_1C_1O$.

So, $\angle BAO=\angle C_1B_1O$ & $\angle CAO=\angle B_1C_1O$

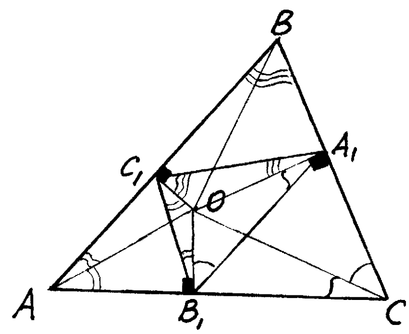


Fig. 18

Similarly, we get one more pair of equalities:

$$\angle CBO=\angle A_1C_1O \quad \& \quad \angle ABO=\angle C_1A_1O$$

$$\angle BCO=\angle A_1B_1O \quad \& \quad \angle ACO=\angle B_1A_1O$$

Hence, by passage from the n-th pedal triangle to the n+1-th pedal triangle we obtain the follows equalities in the form of diagram:

$\angle B_nA_nO$	$\angle C_nA_nO$	$\angle C_nB_nO$	$\angle A_nB_nO$	$\angle B_nC_nO$	$\angle A_nC_nO$
‖	‖	‖	‖	‖	‖
$\angle C_{n+1}B_{n+1}O$	$\angle B_{n+1}C_{n+1}O$	$\angle A_{n+1}C_{n+1}O$	$\angle C_{n+1}A_{n+1}O$	$\angle A_{n+1}B_{n+1}O$	$\angle B_{n+1}A_{n+1}O$

We set that triangles $A_oB_oC_o$ and ABC are equal.

Hence, increasing the indexes by every iteration by the diagram we get

$$\angle B_nA_nO = \angle C_{n+1}B_{n+1}O = \angle A_{n+2}C_{n+2}O = \angle B_{n+3}A_{n+3}O$$

$$\angle C_nA_nO = \angle B_{n+1}C_{n+1}O = \angle A_{n+2}B_{n+2}O = \angle C_{n+3}A_{n+3}O$$

Addition the angles of the both of chains yields $\angle A_n = \angle A_{n+3}$.

Similarly, $\angle B_n = \angle B_{n+3}$ & $\angle C_n = \angle C_{n+3}$, i.e. triangles $A_nB_nC_n$ and $A_{n+3}B_{n+3}C_{n+3}$ are *similar*. In particular, $A_3B_3C_3$ and ABC are similar.

Assume

$$\angle BAO=\alpha', \ \angle CAO=\alpha'', \ \angle ACO=\gamma', \ \angle BCO=\gamma'', \ \angle CBO=\beta', \ \angle ABO=\beta''.$$

Exercise. Prove an analogy of the Cheve's theorem:

For the rays l_A, l_B, l_C with the vertices of the respective vertices of triangle to be intersected at one point O it is necessary and sufficient that

$$\sin\alpha'\cdot\sin\beta'\cdot\sin\gamma' = \sin\alpha''\cdot\sin\beta''\cdot\sin\gamma''$$

where $\alpha', \alpha'', \beta', \beta'', \gamma', \gamma''$ are angles on which those rays divide the angles $\angle A$, $\angle B$, $\angle C$ of triangle, respectively.

Exercise. Prove that the ratio of similitude of triangles ABC & $A_3B_3C_3$ is equal to $\sin\alpha'\cdot\sin\beta'\cdot\sin\gamma'$.

(*Hint. Determine what's the connection between the distance from the point O to the vertices of two consequent pedal triangles*).

PROBLEMS FROM GRADUATE EXAMS. SOLUTIONS.

PROBLEMS FROM GRADUATE EXAMS ON MATHEMATICS (FOR BAGRUT)

PROBLEMS:

Part 1. (10 points for every question. Solve 3 from 5 only).

1. Given right triangle ABD. Through B drawn a circle intersecting hypotenuse at points C and E such that BC is the diameter of circle. Express ED through AB and BD if known that AB=d cм, BD=a cм.

2. a) Prove that $x^2 - 5x + 7$ positive for any real x.
   b) Find all values of x that following inequality holding true

$$(x^2-5x+7)^{x^2-8x+6} > (x^2-5x+7)^{1-2x}$$

3. Prove by mathematical induction method or by any other way that

$$1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + 4\cdot 2^4 + \ldots + (2n+1)\cdot 2^{2n+1} = 2 + n\cdot 2^{2n+3}$$

4. Given an infinite decreasing geometric progression in which all terms are positive. Denote by: S_n the sum of n first terms and S the sum of all progression. $(S-S_n)$ is n-th term of new progression.

a) Prove that the new progression with general term $(S-S_n)$ also the infinite geometric progression.
b) Find the sum of the new progression if $S_2=216$ & $S=243$.

5. We have to elect the municipality commission from 10 men and 6 women. The commission is consist of chairman, his adviser and 3 members. They´re agreed that chairman and his adviser should be of different sex and three of members of the same sex. In how many different ways can the commission be elected ?

Part 2. (2 questions from 5).

6. Given that in isosceles trapezoid ABCD ($AD \parallel BC$) the smallest base is equal to the lateral side. The angle by the greatest base is equal to α. Through the vertex D drawn a line intersecting AB at the point E and formed an angle β with the base AD.
a) express the ratio of areas of triangles to area of trapezoid
b) prove that if given $\alpha=60^o$ & $\beta=30^o$ then the ratio is equal to 2/3.

7. Solve equation:
a) $\sin^2 3x - \sin^2 x = \sin x\cdot\cos x$
b) prove that in an acute triangle $\tan\alpha\cdot\tan\beta > 1$.

8. Given $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$ (α,β,γ are angles of a triangle). Prove that this triangle is right.

9. The base of right prism is an equilateral triangle with the side a. The angle between diagonal of lateral face and another lateral face is equal to α. Express the volume of prism through a and α.

10. Given a line perpendicular to the two lines in the plane which are going through the point of intersection line with the plane. Prove that this line perpendicular to any line going through that point.

Part 3. (2 questions from 3).

11. Side AB of triangle ABC belongs to the line $y=2x$, side AC belongs to the line $y=-4x+12$, the altitude dropped to the side BC belongs to the line $y=x+2$. Given that $BC=\sqrt{32}$. Find coordinates of the triangle. (*There are two ways of solution*).

12. Equation of a circle is $x^2+y^2+6x+5=0$. Find geometric location of all centres of circles which are going through point (3;0) and touching given circle.

13. Through the point $A(x_1, y_1)$ lying on the parabola $y^2=2px$ draw the tangent to the parabola. From the focus F of the parabola drop perpendicular to the tangent. Denote by G the point of intersection the perpendicular with line $x=-\frac{p}{2}$. Prove that the tangent bisect segment FG.

Part 4. (2 questions from 3).

14. Given function $y=\frac{a\cdot e^x}{x+b}$ and known that $x=-2$ is asymptotic of this function. The tangent to this function at the point $x=0$ and positive direction of axis OX formed angle 45^o.

a) Find a and b.
b) Find points of intersection this function with axises.
c) Find domain of increasing and decreasing of function.

15. Had to set cable from electric power station C which located at the river's bank to another river's bank at the point A so that the part of the cable be situated under water and the rest of cable be situated along the the bank. Given

$|AB| = 100$ m

$|CB| = 60$ m

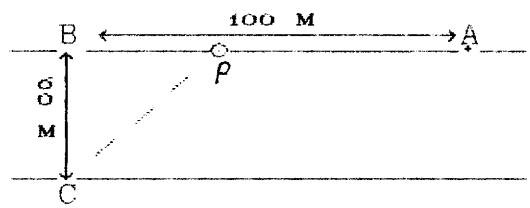


Fig. 1

Setting cable under water coast 130 nis. per meter and along the bank 50 nis. per meter. Find minimal expenses for cable setting.

16. Graphs of functions $y=x^2$ & $y=\sqrt{b^3x}$ $(b>0)$ intersect at point A and at the origin of coordinates. Prove that OA bisect the area between two graphs.

SOLUTIONS:

1.

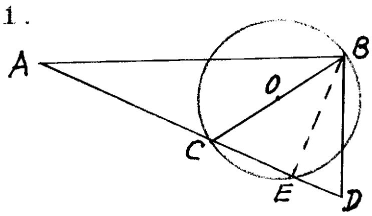


Fig. 2

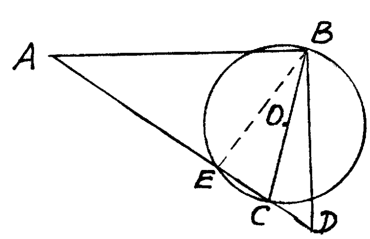


Fig. 3

Pay attention that is possible only two situation shown at those figures 2,3 since in the problem's condition nothing said about the location of points C and F on the hypotenuse. However, it doesn't influence on needed value of ED. Actually, in the both of cases connect B with E. Then $\angle BEC=90^{o}$ as inscribed into circle and leaning on the diameter. But then BE is altitude dropped from the vertex B to hypotenuse, therefore, needed segment is the projection of leg BD on hypotenuse AD. Triangles BED & ABD are similar. Then,

$\frac{ED}{DB}=\frac{BD}{AD}$. Hence, $ED=\frac{BD^2}{AD}=\frac{a^2}{\sqrt{a^2+d^2}}$ sm.

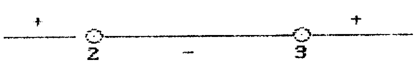
2. a) $x^2-5x+7=(x-\frac{5}{2})^2+7-\frac{25}{4}=(x-\frac{5}{2})^2+\frac{3}{4}\ge\frac{3}{4}>0$

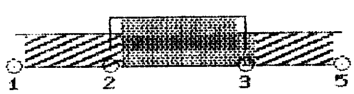
b) $(x^2-5x+7)^{x^2-8x+6}>(x^2-5x+7)^{1-2x}\iff$

$$\iff\left[\begin{array}{l}\begin{cases}x^2-5x+7<1\\ x^2-8x+6<1-2x\end{cases}\\ \begin{cases}x^2-5x+7>1\\ x^2-8x+6>1-2x\end{cases}\end{array}\right.\iff\left[\begin{array}{l}\begin{cases}x^2-5x+6<0\\ x^2-6x+5<0\end{cases}\\ \begin{cases}x^2-5x+6>0\\ x^2-6x+5>0\end{cases}\end{array}\right.\iff\left[\begin{array}{l}\begin{cases}2<x<3\\ 1<x<5\end{cases}\\ \begin{cases}\left[\begin{array}{l}x<2\\ x>3\end{array}\right.\\ \left[\begin{array}{l}x>2\\ x<3\end{array}\right.\end{cases}\end{array}\right.\iff\left[\begin{array}{l}2<x<3\\ \left[\begin{array}{l}x<1\\ x>5\end{array}\right.\end{array}\right.$$

$$\iff\left[\begin{array}{l}x<1\\ 2<x<3\\ 5<x\end{array}\right.\iff x\in(-\infty,1)\cup(2,3)\cup(5,\infty)$$

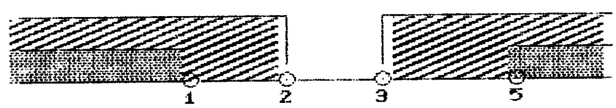
Remark: $x^2-5x+6=(x-2)\cdot(x-3)$; $x^2-6x+5=(x-1)\cdot(x-5)$





$\begin{cases}2<x<3\\ 1<x<5\end{cases}$

Fig. 4



$\begin{cases}\left[\begin{array}{l}x<2\\ x>3\end{array}\right.\\ \left[\begin{array}{l}x<1\\ x>5\end{array}\right.\end{cases}$

Fig. 5

3. *1-st way - Matematical complete induction method:*

1. Base.
   $n=1$. We have $1\cdot 2^1+2\cdot 2^2+3\cdot 2^3=2+8+24=34$.
   From other hand, $2+1\cdot 2^{2+3}=2+2^5=34$.
2. Induction's step $n\to n+1$:

$1\cdot 2^1+2\cdot 2^2+\ldots+(2n+1)\cdot 2^{2n+1}+(2n+2)\cdot 2^{2n+2}+(2n+3)\cdot 2^{2n+3}=$

$=2+n\cdot 2^{2n+3}+(2n+2)\cdot 2^{2n+2}+(2n+3)\cdot 2^{2n+3}=2+n\cdot 2^{2n+3}+(n+1)\cdot 2^{2n+3}+(2n+3)\cdot 2^{2n+3}$

$= 2+2^{2n+3}\cdot(n+n+1+2n+3) = 2+(4n+4)\cdot 2^{2n+3} = 2+(n+1)\cdot 2^{2n+5} =$

$= 2 + (n+1)^{2(n+1)+3}$, what was needed to prove.

2-nd way.

Consider sum $S_n(x) = 1 + x + x^2 + \ldots + x^{2n+1}$ or $S_n(x) = \dfrac{x^{2n+2} - 1}{x - 1}$

Then its derivative $S_n'(x)$ from one hand, is equal to $1 + 2x + \ldots + (2n+1)\cdot x^{2n}$

From other hand, $S_n'(x) = \dfrac{(2n+2)\cdot x^{2n+1}\cdot(x-1) - (x^{2n+2}-1)}{(x-1)^2} =$

$= \dfrac{(2n+1)\cdot x^{2n+2} - (2n+2)\cdot x^{2n+1} + 1}{(x-1)^2} \Rightarrow S_n'(2) = \dfrac{(2n+1)\cdot 2^{2n+2} - (2n+2)2^{2n+1}+1}{(2-1)^2} = n\cdot 2^{2n+2}+1$

$$S_n(2) = 2S_n'(2) = 1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \ldots + (2n+1)\cdot 2^{2n+1} = n\cdot 2^{2n+3} + 2$$

4. $S_n = a_1 + a_2 + \ldots + a_n$. $a_n = a_1\cdot q^{n-1}$, где $|q|<1$.

Then, $S = a_1 + a_2 + \ldots + a_n + \ldots$, i.e. $S = \lim\limits_{n\to\infty} S_n$

a) Show that sequence $S - S_1,\ S - S_2,\ \ldots,\ S - S_n, \ldots$ – also progression.

Actually, $S - S_n = \dfrac{a_1}{1-q} - \dfrac{a_1\cdot(1-q^{n+1})}{1-q} = \dfrac{a_1}{1-q}\cdot q^{n+1} = \dfrac{a_1\cdot q^2}{q-1}\cdot g^{n-1}$

It means that sequence $S - S_1,\ S - S_2,\ \ldots,\ S - S_n,\ \ldots$ is infinite decreasing geometric progression with common ratio q and the first term $\dfrac{a_1\cdot q^2}{1-q}$.

b) Its sum is equal to $\dfrac{a_1\cdot q^2}{1-q}\cdot\dfrac{1}{1-q} = \dfrac{q^2}{1-q}\cdot\dfrac{a_1}{1-q} = \dfrac{q^2}{1-q}\cdot S.$

Since $S_2=216$ & $S = 243$ we have system:

$$\begin{cases} a_1 + a_1\cdot q = 216 \\ \dfrac{a_1}{1-q} = 243 \end{cases} \Leftrightarrow \begin{cases} 1-q^2 = \dfrac{216}{243} = \dfrac{8}{9} \\ a_1 = 243(1-q) \end{cases} \Leftrightarrow \begin{cases} q^2 = \dfrac{1}{9} \\ a_1 = 243(1-q) \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} q = \frac{1}{3} \\ a_1 = 162 \end{cases} \\ \begin{cases} q = -\frac{1}{3} \\ a_1 = 324 \end{cases} \end{array}\right.$$

It follows that problem's condition holding true by two geometric progression with ratios 1/3 and -1/3 respectively.
And therefore, we obtain two answers: the sum of infinite decreasing geometric progression $S - S_1,\ S - S_2,\ \ldots,\ S - S_n,\ \ldots$ can be equal to

$\dfrac{1/9}{1-1/3}\cdot 243 = \dfrac{243}{6} = \dfrac{81}{2}$ in the case of $q=1/3$, and then $a_1=162$, or

$\dfrac{1/9}{1+1/3}\cdot 243 = \dfrac{243}{12} = \dfrac{81}{4}$ in the case of $q=-1/3$, и тогда $a_1 = 324$.

5. All possible staffs of commission are represented by following cases:

1. Chairman is man, adviser is woman and rest of members are women, i.e. 1 man and 4 women.
2. Chairman is woman, adviser is man and rest of members are women, i.e. 1 man and 4 women.
3. Chairman is man, adviser is woman and rest of members are men, i.e. 4 men and 1 woman.
4. Chairman is woman, adviser is man and rest of members are men, i.e. 4 men and 1 woman.

All of those cases can be divided by 2 groups:
Commission of 1-st type - one woman and 4 men.
Commission of 2-nd type - one man and 4 women.

Commission are distinct by the set of men from municipality and by the chairman of commission. For example, from the set $\{w, m_1, m_2, m_3, m_4\}$ we can choose groups electing on the post chairman one of the four men - $\{w, m_1\}$, $\{w, m_2\}$, $\{w, m_3\}$, $\{w, m_4\}$. Actually, it's very important to know who's chairman and who's adviser in the administration pair. Hence, from the one set of men we can make 8 different commission. Since 4 men from 10 we can chose by $\binom{10}{4}$ ways and such set we can add one woman by 6 ways. Finally, the number of sets of the 1-st type is $6\cdot\binom{4}{10}$ and number of comissions is $8\cdot 6\cdot\binom{10}{4}$. So, the number of possible election of commissions is

$$8\cdot\left(6\cdot\binom{10}{4} + 10\cdot\binom{10}{4}\right) = \frac{8\cdot 6\cdot 10\cdot 9\cdot 8\cdot 7}{4} + \frac{8\cdot 10\cdot 6\cdot 5}{1\cdot 2} = 20\cdot 9\cdot 56 + 8\cdot 25\cdot 6 = 12080$$

(Similarly, number of commissions of the 2-nd type is $8\cdot 10\cdot\binom{6}{4}$).

6.

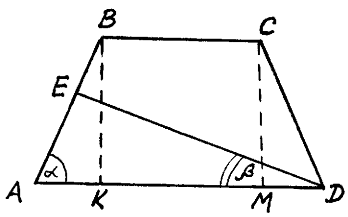


Fig. 6

Denote the length of BC (*fig. 6*) by a. Then AB=BC=CD=a. Drop from the vertex B & C on side AD perpendiculars BK & CM. Then KM=BC as opposite sides in rectangle and AK=MD as projection onto AD equals and with same angle to AD. Therefore,

$AD = BC + 2\cdot AK = a + 2\cdot a\cdot\cos\alpha$. Hence,

$$BC + AD = 2a\cdot(1 + \cos\alpha) \;\&\; BK = a\cdot\sin\alpha \Rightarrow S_{ABCD} = \frac{BC + AD}{2}\cdot BK =$$

$$= \frac{2a^2\cdot(1+\cos\alpha)\cdot\sin\alpha}{2} = a^2\cdot(1 + \cos\alpha)\cdot\sin\alpha.$$

For determination the area of triangle AED we use the formula for the area of triangle through two angles and side lying by them: $S_{AED} = \frac{AE\cdot ED}{2}\sin\angle AED$

$$S_{AED} = \frac{AE\cdot ED\cdot\sin(\alpha+\beta)}{2}.$$ But by the sine theorem: $$\frac{AE}{\sin\beta} = \frac{DE}{\sin\alpha} = \frac{AD}{\sin(\alpha+\beta)}$$

Thus, $AE = \frac{AD\cdot\sin\beta}{\sin(\alpha+\beta)}$ & $DE = \frac{AD\cdot\sin\alpha}{\sin(\alpha+\beta)}$ and it follows $S_{AED} = \frac{AD^2\sin\alpha\cdot\sin\beta}{2\sin(\alpha+\beta)}$

Since $AD = a + 2a\cdot\cos\alpha$, we get $$\frac{S_{AED}}{S_{ABCD}} = \frac{a^2(1+2\cos\alpha)^2\cdot\sin\alpha\cdot\sin\beta}{2\cdot\sin(\alpha+\beta)\cdot a^2\cdot(1+\cos\alpha)\cdot\sin\alpha} =$$

$$= \frac{(1 + \cos\alpha)^2 \cdot \sin\beta}{2\cdot\sin(\alpha+\beta)\cdot(1+\cos\alpha)}.$$

b) Let $\alpha = 60^\circ$ & $\beta = 30^\circ$, then $\frac{S_{AED}}{S_{ABCD}} = \frac{(1+2\cos 60^\circ)^2\cdot\sin 30^\circ}{2\cdot\sin 90^\circ\cdot(1+\cos 60^\circ)} = \frac{2}{3}$.

7. a) $\sin^2 3x - \sin^2 x = \sin x\cdot\cos x \Leftrightarrow (\sin 3x - \sin x)\cdot(\sin 3x + \sin x) =$

$\sin x\cdot\cos x \Leftrightarrow 2\cdot\cos 2x\cdot\sin x\cdot 2\cdot\sin 2x\cdot\cos x = \sin x\cdot\cos x \Leftrightarrow$

$2\cdot\sin 4x\cdot\sin 2x = \sin 2x \Leftrightarrow \sin 2x\cdot(2\cdot\sin 4x - 1) = 0 \Leftrightarrow$

$$\Leftrightarrow \left[\begin{array}{l}\sin 2x = 0\\ \sin 4x = 1/2\end{array}\right. \Leftrightarrow \left[\begin{array}{l}2x = k\cdot\pi\\ 4x = (-1)^n\cdot\frac{\pi}{6} + n\cdot\pi\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x = \frac{k\cdot\pi}{2},\\ x = (-1)^n\cdot\frac{\pi}{24} + \frac{n\pi}{4}\end{array}\right. k\in\mathbb{Z}, n\in\mathbb{Z}$$

b) We have $\alpha+\beta+\gamma = \pi$ & $0<\alpha,\beta,\gamma<\frac{\pi}{2}$. Moreover, $\alpha+\beta\neq\frac{\pi}{2}$ otherwise $\gamma=\frac{\pi}{2}$.
Then, $\tan(\alpha+\beta) = \tan(\pi-\gamma) = -\tan\gamma$. From other hand:

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}$. Отсюда, $-\tan\gamma = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta} \Leftrightarrow$

$\Leftrightarrow 1 - \tan\alpha\cdot\tan\beta = -\frac{\tan\alpha + \tan\beta}{\tan\gamma}$. Since $\tan\alpha, \tan\beta, \tan\gamma > 0$, then finally we get $\tan\alpha\cdot\tan\beta - 1 > 0 \Leftrightarrow \tan\alpha\cdot\tan\beta > 1$.

8. Transform expression $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1$ which is equal to 0.
We have $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 2\cos(\alpha+\beta)\cdot\cos(\alpha-\beta) + 2\cos^2\gamma - 1 + 1 =$

$$= 2\cdot\cos(\pi-\gamma)\cdot\cos(\alpha-\beta) + 2\cdot\cos^2\gamma = 2\cdot\cos\gamma\cdot(\cos\gamma - \cos(\alpha-\beta)) =$$

$$= 4\cdot\cos\gamma\cdot\sin\frac{\gamma+\alpha-\beta}{2}\cdot\sin\frac{\alpha-\beta-\gamma}{2}.$$

But $\alpha+\beta+\gamma = \pi$, therefore $\frac{\gamma+\alpha-\beta}{2} = \frac{\pi-2\beta}{2}$ & $\frac{\alpha-\beta-\gamma}{2} = \frac{2\alpha-\pi}{2}$. Hence,
$0 = 1 + \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4\cdot\cos\alpha\cdot\cos\beta\cdot\cos\gamma$.
Thus, either $\cos\alpha = 0$ or $\cos\beta = 0$ or $\cos\gamma = 0$, i.e. one of three angles α,β,γ must be equal to $\pi/2$, since $0<\alpha,\beta,\gamma<\pi$.

We could solve the problem otherwise, by formula:

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cdot\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\gamma}{2}\cdot\cos\frac{\beta+\gamma}{2},$$

($\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) =$

$= (\cos\alpha + \cos\beta) + (\cos\gamma + \cos(\alpha+\beta+\gamma)) = 2\cdot\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} + 2\cdot\cos(\gamma+\frac{\alpha+\beta}{2})\cdot$

$\cos\frac{\alpha+\beta}{2} = 2\cdot\cos\frac{\alpha+\beta}{2}\cdot(\cos\frac{\alpha-\beta}{2} + \cos(\gamma+\frac{\alpha+\beta}{2})) = 4\cdot\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\gamma}{2}\cdot\cos\frac{\beta+\gamma}{2}$)

By the way, we can get similarly formula for sines:

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\cdot\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\gamma}{2}\cdot\sin\frac{\beta+\gamma}{2}.$$

Those formulas can be useful in the problems about angles of triangle like we have now. Actually, since $\alpha+\beta+\gamma=\pi$

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+1=\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos(2\alpha+2\beta+2\gamma)=4\cos\alpha\cdot\cos\beta\cdot\cos\gamma$$

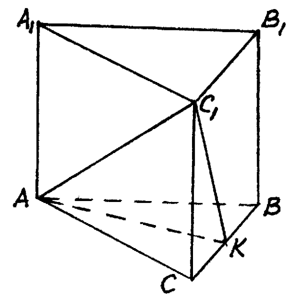


Fig. 7

9. $AB = a$. Since angle between a line and the plane by definition is the angle between the line and its projection onto the plane. We have to drop projection of diagonal of lateral face onto another lateral face (see *fig.* 7). We take diagonal AC_1 and project it onto face CBB_1C_1. (Choice of lateral face is not unique, however since we have a right prism and in the base lying equilateral triangle, then all pairs of diagonal and lateral face are equal accurate to the location in the space by geometric sense).
Since the prism is right then the perpendicular dropped from the point A on the plane CBB_1C_1 is entirely lie in plane ABC which is perpendicular to CBB_1C_1. Moreover, point K of intersection of this perpendicular with plane CBB_1C_1 simultaneously belongs to the planes ABC & CBB_1C_1, i.e. point lies on the line CB of intersection those planes, and it means K is a feet of altitude drawn from the vertex A on side BC of triangle ABC, i.e. midpoint of this side (since triangle ABC is equilateral). $\angle AC_1K = \alpha$ by problem condition.
Now we can compute the volume:

$$V_{priz} = S_{ABC}\cdot CC_1.\quad S_{ABC} = \frac{a^2\cdot\sin 60^0}{2} = \frac{a^2\cdot\sqrt{3}}{4},\quad \frac{AK}{C_1K} = \tan\alpha \quad \& \quad AK = \frac{a\sqrt{3}}{2}.$$

Hence, $C_1K = \frac{a\sqrt{3}}{2}\cdot\cot\alpha$. But since $CK = \frac{a}{2}$, then $CC_1 = \sqrt{C_1K^2 - CK^2} =$

$$= \sqrt{\frac{3a^2}{4}\cdot\cot^2\alpha - \frac{a^2}{4}} = \frac{a}{\sin\alpha}\cdot\sqrt{\left(\frac{\sqrt{3}}{2}\cdot\cos\alpha - \frac{1}{2}\cdot\sin\alpha\right)\cdot\left(\frac{\sqrt{3}}{2}\cdot\cos\alpha + \frac{1}{2}\cdot\sin\alpha\right)} =$$

$$= \frac{a}{\sin\alpha}\cdot\sqrt{\sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}+\alpha\right)}$$. (*Remark*: $C_1K > CK \Leftrightarrow \frac{a\sqrt{3}}{2}\cot\alpha > \frac{a}{2}$ $\Leftrightarrow \cot\alpha > \frac{1}{\sqrt{3}} \Leftrightarrow \alpha < \frac{\pi}{3}$). Thus, we obtain the volume of prism

$$V = \frac{a^2\sqrt{3}}{4}\cdot\frac{a}{\sin\alpha}\cdot\sqrt{\sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}-\alpha\right)} = \frac{a^3}{4\sin\alpha}\cdot\sqrt{3\cdot\sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}+\alpha\right)}$$

10. Denote by
a, b two lines intersection at point O,
c an arbitrary line in the plane π,
l line perpendicular to lines a & b, and going through point O. (*fig. 8*).

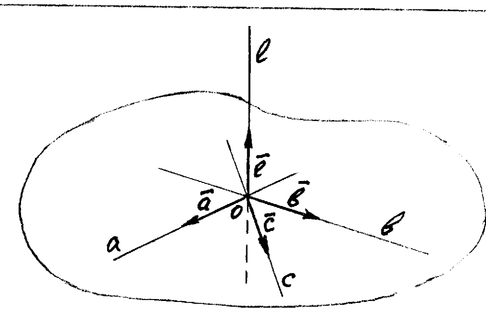


fig. 8

Denote by $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{l}$ are direction vectors of respective lines.
Then by condition, the inner (scalar) product $(\overline{l},\overline{a}) = 0$ & $(\overline{l},\overline{b}) = 0$. Since $\overline{a}$ & $\overline{b}$ by condition are not collinear, then for $\overline{c}$ there exist real k and m such that $\overline{c} = k\cdot\overline{a} + m\cdot\overline{b}$. Hence, $(\overline{l},\overline{c})=(\overline{l},k\cdot\overline{a}+m\cdot\overline{b})=(\overline{l},k\cdot\overline{a}) + (\overline{l},m\cdot\overline{b}) =$
$= k\cdot(\overline{l},\overline{a}) + m\cdot(\overline{l},\overline{b}) = 0$, i.e. lines l & c are perpendicular.

It's possible a geometric proof.
Take on c an arbitrary point distinct from O, denote its by D (*fig. 9*). Then through D we can draw segment with the ends on lines a & b which bisect by point D. (On the line taken point D_1 symmetric to O relatively D and draw $D_1A \parallel b$ & $D_1B \parallel a$, then OAD_1B is parallelogram AB its diagonal and bisect by point D).

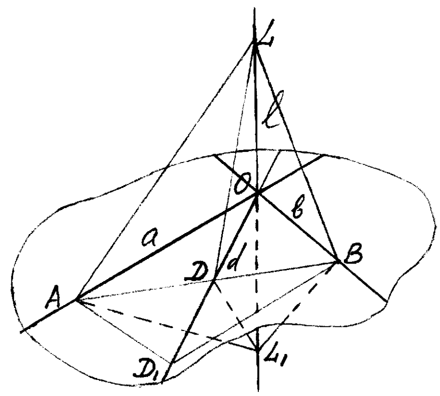


fig. 9

Take on l two mutually symmetric points L & L_1 relatively O. Connect them with points A, B & D.
Then by equality of triangle OLA and OL_1A we get. Similarly, $LB=L_1B$. Then triangles LAB & L_1AB are equal by three sides. And therefore medians LD and L_1D are equal.
From equality of triangle OLD & OL_1D follows equality of angles $\angle DOL=\angle DOL_1$, therefore, l perpendicular to d.

11. Determine coordinates of vertex A (*fig. 10*).

$$\begin{cases} y = -4x + 12 \\ y = 2x \end{cases} \Leftrightarrow \begin{cases} y=2x \\ 6x=12 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = 4. \end{cases}$$

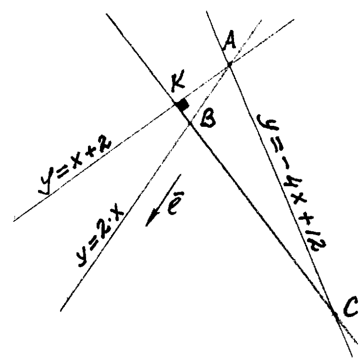


Fig. 10 (schematic, but sufficiently concordant with given data).

Point A lies on line y=x+2 and we can use for control of derived coordinates of the point A.
So, A(2,4). Side BC lies on line directed vector which is normal vector for line
$y=x+2 \Leftrightarrow x-y+2=0$
So, direction vector of line BC is equal to $\overline{e}(1,-1)$. Hence, if we denote coordinates $B(x_1,y_1)$ & $C(x_2,y_2)$, then $\overline{BC}(x_2-x_1,y_2-y_1)$ collinear $\overline{e}$, i.e. there exist $t \in \mathbb{R}$ such that $x_2-x_1=t;\ y_2-y_1=-t$.

Since $|BC|=\sqrt{32}$, hence $BC^2=32=(x_2-x_1)^2+(y_2-y_1)^2=2t^2 \Leftrightarrow t^2=16 \Leftrightarrow \left[\begin{array}{l} t=4 \\ t=-4 \end{array}\right.$

From other hand $B(x_1,y_1)$ lies on y=2x, hence $y_1=2x_1$ and because $C(x_2,y_2)$ lies on $y=-4x+12$, thus, $y_2=-4x_2+12$. Now convenient to express coordinates through t and get final values by substituting in derived expressions the values of t We have $y_2 = y_1-t = 2x_1-t$, $x_2=x_1+t$. Substitution x_2 & y_2 into $y_2=-4x_2+12$, yields

$2x_1-t = -4x_1-4t+12 \iff 6x_1=-3t+12 \iff 2x_1=-t+4$. Hence, $\begin{cases} x_1=\frac{4-t}{2} \\ y_1 = 4-t \end{cases}$ and

$\begin{cases} x_2= x_1+ t = \frac{4-t}{2} + t = \frac{4+t}{2} \\ y_2= 2x_1- t = 4-t-t = 4 - 2t. \end{cases}$ Thus, when $t=4$ $B(0,0)$; $C(4,-4)$ when $t=-4$ $B(4,8)$; $C(0,12)$

12. $x^2 + y^2 + 6x + 5 = 0 \iff (x+3)^2+y^2=4$.
$R = 2$, $A(-3,0)$; $B(3,0)$. $AB = 6 > 2 = R$, therefore, point B lies outside given circle.

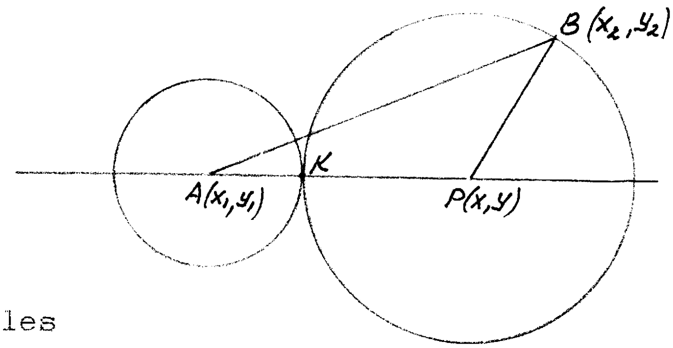


Fig. 11

Let $A(x_1,y_1)$ be centre of given circle, R its radii, $B(x_2,y_2)$ given point.
Let $P(x,y)$ be the centre of circle which touch given circle at the point K and going through given point B. Radii drawn to the point of touching of the circles lie on line connecting the centres of those circles. Possible two cases of mutually location of circles (see *fig. 11, 12*).

Outside touch (*fig. 11*), then $AK + KP = AP \iff AK=AP-BP \iff R=PA-PB$, and inside touch (*fig. 12*), then $PA = PK - AK \iff R=PB-PA$. By union those cases we get: $R = |PA - PB|$. Therefore, point P belongs to the set of points such that absolute value of difference of the distance from each of them to the two given points is a constant value R. It means that P belongs to hyperbola with focuses at the points A & B.

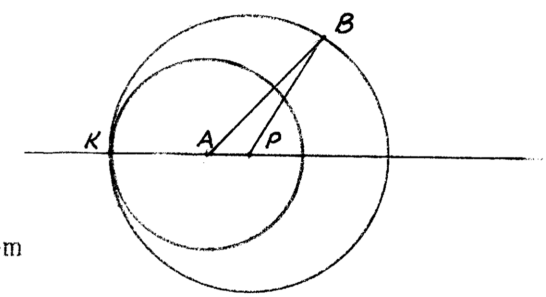


Fig. 12

From other hand, let point P be lying on hyperbola with focuses at points A & B and $|PA-PB|=R$, then if we draw a circle with centre at point P and radius PB, then either $PA=PB+R$ or $PA=PB-R$. Connect centers P & A. If $PA=PB+R$, then by taking on PA point K on distance R from A we get that K lies on constructed circle and $PA=AK+KB$ that's possible only in the case of touching. If $PA=PB-R$, then on continuation PA in the direction from P to A by length R we get point K for which $AK=R$, i.e. K lies on given circle and $PK=PA+AK=PA+R=PB-R+R=PB$. And it means that K lies on the constructed circle and on the line connecting centres of the both circles that's possible only in the case of touching. So, we've proved that needed geometric location of all centres is hyperbola. Now we can write its equation. The focuses of hyperbola A & B are situated by problem's condition and symmetric relatively origin of coordinates X-axis (*fig. 13*).

Equation of the hyperbola in this case is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $2a=R$ (here $\Leftrightarrow a=1$) & $b=\sqrt{c^2-a^2}$

But $c=3$. Hence, $b=\sqrt{9-1}$ and equation can be written $\frac{x^2}{1} - \frac{y^2}{8} = 1$.

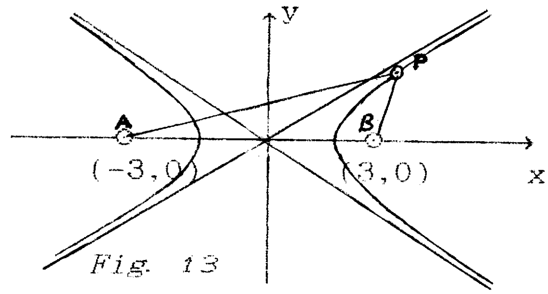


Fig. 13

Verify: $2 = |PA - PB|$, $y^2=8(x^2-1)$.

$PA^2 = (x+3)^2 + y^2 = (x+3)^2 + 8\cdot(x^2-1) = 9x^2 + 6x + 1 = (3x+1)^2$

$PB^2 = (x-3)^2 + y^2 = (3x-1)^2$.

Hence, $|PA - PB| = ||3x+1| - |3x-1||$, and $|x|\geq 1$. But then if $x>1$, then $|3x+1| - |3x-1| = 3x+1-3x-1 = 2$. If $x<-1$, then $|3x+1|=-3x-1$ & $|3x-1|=1-3x$ and then $|3x+1| - |3x-1| = -2$.

13. Line $x=-\frac{p}{2}$ is directrix of parabola

So, let G be a point of intersection of perpendicular of the tangent to parabola at $A(x_1, y_1)$ (*fig. 14*). Equation of the tangent to parabola $y^2=2px$ at point $A(x_1, y_1)$ is $yy_1=p\cdot(x+x_1)$ or $px - yy_1 + px_1 = 0$. Hence, $\overline{e}(p, -y_1)$ is normal vector of the tangent and it follows that it's direction vector of a line perpendicular to the tangent.

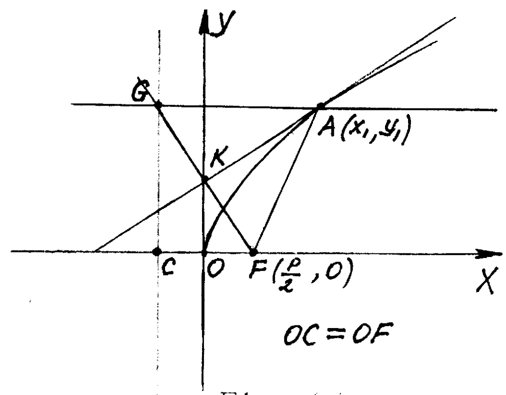


Fig. 14

Equation of perpendicular to the tangent can be written in parametric form since normal vector of tangent $(p, -y_1)$ is a direction vector for perpendicular to the tangent dropped from point $F(p/2, 0)$. We have:

$$\begin{cases} x = \frac{p}{2} + pt \\ y = -y_1 t, \ t\in\mathbb{R} \end{cases}$$

. Find coordinates of the point G if given that G lies on directrix and on the perpendicular to the tangent. We get

$-\frac{p}{2} = \frac{p}{2} + pt \iff t=-1$ & $\begin{cases} x=-\frac{p}{2} \\ y=y_1 \end{cases}$, i.e. point G lies on a line drawn through A and parallel to axis OX. Let K be a midpoint of segment FG, then its coordinates

$\frac{-\frac{p}{2} + \frac{p}{2}}{2} = 0$ & $\frac{y_1+0}{2} = \frac{y_1}{2}$, or $K(0, \frac{y_1}{2})$. Show that this point lies on the tangent and on perpendicular to it dropped from F

Actually, K lies on the tangent, because $y_1 \cdot \frac{y_1}{2} = P(0+x_1) \iff y_1^2 = 2px_1$.

From the other hand, in equation of perpendicular $t=-1/2$ we get the coordinates of point K. What's needed to prove.

It's possible another solution by property of parabola's focus. The line ℓ drawn through point A of parabola is parallel to axis OX and formed with tangent of parabola at point A the same angle as a line drawn from the focus F in point of touching. (*Fig. 15*)

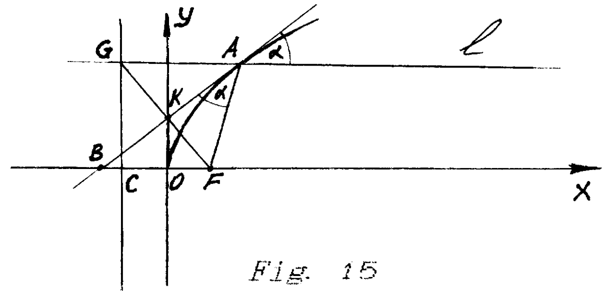


Fig. 15

Let point G be the point of intersection ℓ with directrix. Base property of a parabola (determining) - *the distance from the focus to the point on a parabola is equal to the distance from this point to directix,* i.e. $AG=AF$

Therefore, triangle GAF is isosceles. Since angle $\angle GAB = \alpha$, then the tangent is the bisector of angle $\angle GAF$. Let K be point of intersection tangent with GF. Since bisector in an isosceles triangle is median and simultaneously altitude, then FG is perpendicular to the tangent and bisecting by tangent.

14. Since $x=-2$ is asymptotic to the graph of function $y=\frac{ae^x}{x+b}$, then $b=2$. Actually, if $a\neq 0$, then $x=-b$ is vertical asymptotic. If $a=0$, then function is equal to zero on domain of definition and function has no vertical asymptotic. Since given that $x=-2$ is a vertical asymptotic, then $b=2$.

Determine a by using second condition

$$f(x) = \frac{ae^x}{x+2}, \quad f'(x) = a\cdot\frac{e^x(x+2)-e^x}{(x+2)^2} = \frac{ae^x\cdot(x+1)}{(x+2)^2} \Rightarrow f'(0)=\frac{a}{4}=\tan 45^\circ = 1 \Rightarrow$$

$$a = 4 \Rightarrow y=\frac{4e^x}{x+2}$$

b) Axis OX and the graph of function (*fig. 16*) have no points of intersection, i.e. $y=0 \iff e^x=0$ that's impossible. Axis OY and graph has points of intersection, since when $x=0$ we get $y=2$. So, point $K(0,2)$ is the point of intersection graph with axis OY.

c) $f'(x)=\frac{4e^x(x+1)}{(x+2)^2}$. $\quad f'(x) > 0 \iff x > -1$

$f'(x) < 0 \iff \left[\begin{array}{l} -2 < x < -1 \\ x < -2. \end{array}\right.$

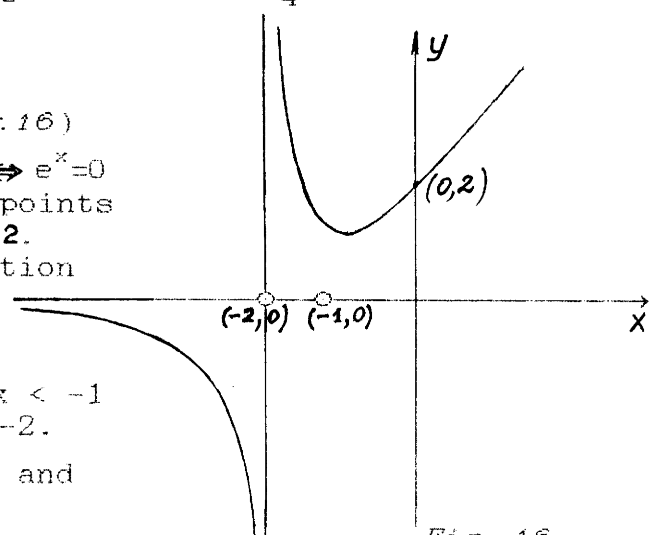


Fig. 16

So, $f(x)$ is monotone increasing on $(-1,\infty)$ and monotone decreasing on each of intervals $(-2,-1)$ & $(-\infty,-2)$. Axis OX is horizontal asymptotic when $x\to-\infty$ since $\lim_{x\to-\infty}\frac{4e^x}{x+2} = 0$.

15. The price of work depend from the location of point P (see *fig 17*). Set $BP=x$. Then the location of P determinate by $x\in[0,100]$ and this location is unique. The price of work can be written

$$|CP|\cdot 130 + |PA|\cdot 50 = 10\cdot((100-x)\cdot 5 + 13\cdot\sqrt{60^2+x^2}).$$

Convenient to introduce new variable t, such that $x=10\cdot t$. Then the price of work is $100((10-t)\cdot 5+13\cdot\sqrt{36+t^2})=100(13\cdot\sqrt{36+t^2}-5t+50)$.

For seek $t\in[0,10]$ that minimize the price of work sufficient to find the value $t\in[0,10]$ which minimize function $f(t)=13\cdot\sqrt{36+t^2}-5t$.

$$\begin{cases} f'(t)=\dfrac{13\cdot t}{\sqrt{36+t^2}}-5=0 \\ t\in[0,10] \end{cases} \Leftrightarrow \begin{cases} 13\cdot t=5\cdot\sqrt{36+t^2} \\ t\in[0,10] \end{cases} \Leftrightarrow \begin{cases} 169t^2=25\cdot 36+25t^2 \\ t\in[0,10] \end{cases} \Leftrightarrow \begin{cases} 144t^2=900 \\ t\in[0,10] \end{cases}$$

$\Leftrightarrow t=\frac{5}{2}$. So, $t=5/2$ is critical point. $\min\limits_{[0,10]} f(t)=\min\{f(0),f(10),f(\frac{5}{2})\}$

$f(0)=13\cdot 6=78$; $f(10)=13\cdot\sqrt{136}-50$; $f(5/2)=13\cdot\sqrt{36+\frac{25}{4}}-\frac{25}{2}=\frac{169}{2}-\frac{25}{2}=72$.

$f(10)>72 \Leftrightarrow 13\cdot\sqrt{136}>122$. Hence, $\min\{f(0),f(10),f(5/2)\}=72=f(5/2)$.

It means that the price of work can be minimal when point P situated on the distance $x=10\cdot 5/2=25$ m. from the point B. In this case the price is equal to $100\cdot(72+50)=12200$ nis.

16. $\begin{cases} y=x^2 \\ y=\sqrt{b^3x} \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ x^4=b^3x \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=0 \\ y=0 \end{cases} \\ \begin{cases} x=b \\ y=b^2 \end{cases} \end{array}\right.$

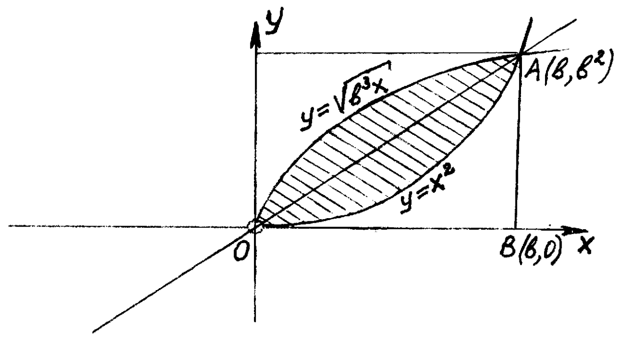


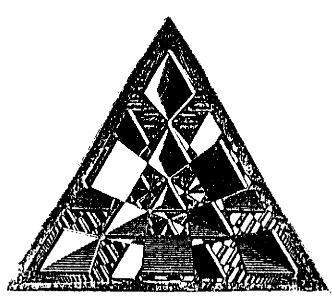
Fig. 18

Compute the area of whole noted figure. (see *fig. 18*).

$$S=\int_0^b(\sqrt{b^3x}-x^2)dx=\left(\sqrt{b^3}\cdot\frac{2}{3}-\frac{x^3}{3}\right)\Big|_0^b=\frac{b^3}{3}$$

The area of triangle OBA easily determine $S_{OBA}=\frac{b^3}{2}$. The area of curvilinear trapezoid bounded by graph $y=x^2$ is equal to $\int_0^b x^2dx=\frac{b^3}{3}$. Thus, the area of noted figure situated under segment OA is equal to $\frac{b^3}{2}-\frac{b^3}{3}=\frac{b^3}{6}$. ■

(The article has been prepared by **Alt Arkady**, "**Blikh**" school, **Ramat-Gan**).



"DELTA"S PROBLEMS

Problems for school:

12-th form:

1. Solve inequality

$$\frac{2^{x+1}-7}{x-1} < \frac{10}{3-2x}$$

2. Prove that for any $x,y,z \geq 0$: $\min\{1, x^3, y^4, z^5\} \leq xyz$.

3. Prove inequality

$$\frac{\log_3 4 \cdot \log_3 6 \cdot \ldots \cdot \log_3 80}{\log_3 3 \cdot \log_3 5 \cdot \ldots \cdot \log_3 79} > 2$$

4. Let m_a, m_b, m_c be a medians of a triangle drawm to theides a,b,c respectively, with $b \neq c$ & $b \cdot m_b = c \cdot m_c$. Prove that

$$\frac{m_b}{c} = \frac{m_c}{b} = \frac{m_a}{a} = \frac{\sqrt{3}}{2}$$

5. Find all values of real parameter $p \in \mathbb{R}$ for which any real number x satisfy at least to one of inequalities

$$x^2-(1+4p)x+4p \geq 0 \ , \ x^2-px-3x+3p < 0.$$

11-th form:

1. Given angles of a triangle. Find the angle between median and bisector.

2. Solve equaiton

$$\frac{tg^2 x + tg^2 y}{1 + tg^2 x + tg^2 y} = \sin^2 x + \sin^2 y$$

3. Find values of parameter b such that system

$$\begin{cases} x \geq (y-b)^2 \\ y \geq (x-b)^2 \end{cases}$$ has a unique solution.

4. Numbers x,y,z such that $x^2+3y^2+z^2=2$. Find a greatest value of expression $2x+y-z$.

5. Solve for integer x,y,z

$$\log_2(2x-3y+5z+1) + \log_2(5y-3x-2z-2) + \log_2(x-2y-3z+4) > z^2 + 9z - 15$$

10-th form:

1. In a triangle with the sides a,b,c drawn the bisectors. The points of intersection bisectors with opposite sides formed a second triangle. Prove that the ratio of triangles areas is equal to

$$\frac{2abc}{(a+b)\cdot(b+c)\cdot(c+a)}$$

2. Prove that $x^{12} - x^9 + x^4 - x + 1 > 0$ for all x.

3. Given

$$\begin{cases} x + y + z = 2 \\ xy + xz + yz = 1 \end{cases}$$ Prove that $x,y,z \in [0,4/3]$.

4. What's greater 2^{57} or 3^{34}.

5. Function $y=a|x| + b|x-k|$ turns to zero if $x=-1$ & $x=3$. Moreover, given that function has a greatest value 2. Find constants a, b, k.

9-th form:

1. Find a value of expression

$$\frac{x+y}{z+t} + \frac{y+z}{t+x} + \frac{z+t}{x+y} + \frac{t+x}{y+z} \quad \text{if} \quad \frac{x}{y+z+t} = \frac{y}{z+t+x} = \frac{z}{t+x+y} = \frac{t}{x+y+z}$$

2. In triangle ABC point K bisect the side BC as (BK:KC=) 1 :2 and point M bisects the side AB as 2:3. (AM:MB = 2:3) Drawn the lines AK & CM which intersect at point E. What's a part of area of the triangle ABC contain the area of triangle KME ?

3. Solve system of equation

$$\begin{cases} x^2 - 3xy = x - 6 \\ y^2 + xy = 8 - y \end{cases}$$

4. Find the smallest value of expression x^2+y^2 if $x+2y=1$.

5. Given that a+b+c=0, where a,b,c are integer. Prove that $2a^4 + 2b^4 + 2c^4$ is an exact square of some integer.

(Problems have been collected by Alt Arcady, "Blikh" school, Ramat-Gan).

"DELTA"'S LABORATORY

In this part of mathematical collection we suggest reader to carry out a mini-research by questions offered below. If the reader has own suggestion, hypothesis by the matter of those question or an associated questions he of course free to carry this research out corresponding to the own tastes and interests. And put into shapes the results what came out as an article the reader could offer that to the other readers through our mathematical collection. It's possible variants with editions, additions, extensions and remarks. Any material of research character or about "Delta"'s laboratory ought send to our editorial office address.
In this issue we offer theme:

ROUND ARROUND CUBIC POLYNOMIAL

Alt Arkady
"Blikh" school
Ramat-Gan

The object of our mini-research is a polynomial of third degree:
$P(x) = a\cdot x^3 + b\cdot x^2 + c\cdot x + d$ and corresponding to its cubic equation:
$a\cdot x^3 + b\cdot x^2 + c\cdot x + d = 0$.

Notice at the beginning that except reducing to unitary form (coefficient by x^3 is equal to 1) by straightforward division equation $ax^3+bx^2+cx+d=0$ by a, we can get equation equivalent to the given by another way namely, by straightforward multiplication equation by a^2 and substitution $ax=t$ yields

$$ax^3 + bx^2 + cx + d = 0 \iff \begin{cases} t^3 + bt^2 + act + a^2d = 0 \\ x = \dfrac{t}{a} \end{cases}$$

Such reducing usually use for searching rational roots where a,b,c,d are integer. In the particular case when d=1, obviously x=0 is not a root of the equation and we can carry out reducing to unitary form by substitution $1/x = t$. In what follows we suppose that the cubic equation get form (i)

$$x^3 + r\cdot x^2 + p\cdot x + q = 0 \qquad (1)$$

1. Prove that equation (1) always has real roots (at least one) (*Show the pair of numbers* m & M *such that* $f(m)\cdot f(M) < 0$. *Then by continuity of function* $f(x)=x^3+rx^2+px+q$ *onto interval* (m,M) *can be found point* c *such that* $f(c)=0$).

2. a) Prove that for any $a \in \mathbb{R}$ $f(x) = (x-a)\cdot g(x) + f(a)$, where $g(x)$ - a quadratic polynomial (*Which ?*). (*It's prticular case of Besu's theorem*).

 b) Prove that equation (1) has no more than three distinct roots.

3. Let $x = a$ be a root, then $f(x)=(x-a)\cdot g(x)$, where $g(x)$ is a quadratic polynomial. If $g(a)\neq 0$, then we shall say that $x=a$ is *simple root* of equation $f(x)=0$, otherwise $x=a$ call *multiple root*. And here's possible if $g(a)=0$ two cases either $g(x)=(x-a)^2$ or $g(x)=(x-a)\cdot(x-b)$, where $b\neq a$. In first case a has multiplicity 3 and $f(x)=(x-a)^3$, in second case a has multiplicity 2 and $f(x)=(x-a)^2\cdot(x-b)$.

 Prove that for any cubic equation of type (1) in $\mathbb{R}$ it's possible either cases:
 1. $f(x)=0$ has an one simple roots and has no other roots in $\mathbb{R}$.
 2. $f(x)=0$ has three simple roots.
 3. $f(x)=0$ has an one root of multiplicity 2 and one simple.
 4. $f(x)=0$ has one root of multiplicity 3.

 Find an examples for each of cases. It would be interesting to find conditions to which satisfy coefficients r, p, q of equation in each of cases.

If we're saying that equation $f(x)=0$ has all root in R, then we mean any of cases except the case when equation has only one simple root.

4. Let all of three root of equation (1) be real. Denote them by x_1, x_2, x_3 (among them could be equal). Then occur equality:

$$x^3 + r\cdot x^2 + p\cdot x + q = (x-x_1)\cdot(x-x_2)\cdot(x-x_3), \text{ and}$$

$$\begin{cases} x_1 + x_2 + x_3 = -r \\ x_1x_2 + x_2x_3 + x_1x_3 = p \\ x_1x_2x_3 = -q \end{cases} \quad (2).$$

(*Wiette's theorem for a polynom of third degree*). Prove it.

(ii). Before going on we notice that complete equation (1) can be reduced by substitution $x = y - \frac{r}{3}$ to the form $y^3 + b\cdot y + c = 0$ (2), without the term by second degree. Derive the formulas a,b,c through p, q, r.

Now we consider more detail this equation.

1. Show that if $b>0$ then equation (2) has a unique real root which by fixed b can be considered as a function $x(c)$ of free term c. And if $c>0$ then this root is negative.

In supposition of 1:

2. Prove that $x(c)$ is monotone decreasing onto $(-\infty,\infty)$.
3. Prove that $x(c)$ is continuous onto $(-\infty,\infty)$.
4. Prove that $x(c)$ is diffirentiable onto $(-\infty,\infty)$ and find the form of derivative $x'(c)$.
5. Prove that $x(c)$ has a derivative of second degree. Find its form.
6. Prove if $c<0$ function $x(c)$ is convex, then when $c>0$ the function is concave and $c=0$ is the point of inflection.

(iii). Now go back to the question of computing the roots of equation (2). Since the case $b=0$ is trivial indeed (by the way, what we can say about roots of equation (2) in this case ?), then we consider two other cases

Case 1. b > 0.
Substitution $x=\sqrt{\frac{b}{3}}\cdot(t-\frac{1}{t})$ into (2) yields quadratic equation relatively t^3. Which ?

1. Is its always solvable ?
2. Does it equivalent to the given equation ?
3. Does appear strange roots ? If yes then how is possible to remove them?

Case 2. b < 0.
Substitution $x=\sqrt{\frac{b}{3}}\cdot(t+\frac{1}{t})$ yields to quadratic equation relatively t^3.
In this case is possible situations when quadratic equation has no roots. (Show it. What situation is it ?). What supplementary condition does necessary to introduce on b and c in order to this way will give a results ? What would we do with two roots of the quadratic equation ? Are there a strange roots, if yes than how can we remove them ? How can we find the rest of the roots of cubic equation ? (of course if there there exist). Are there a way out from the situation when the substitution cannot be used ? Maybe we should transfer to the complex domain ?

(But without transfering into complex case we can get exhaustive information about roots of equation $y^3 + by + c = 0$ *in the set of real numbers. Who wish to get this information we offer following plan (it's not a unique plan) allowing us avoid complicate things which appear by substitutions described before.*

1. Consider cubic equation

$$4\cdot t^3 - 3t = d, \textit{ where } |d| \le 1.$$

Set $d = \cos\alpha$ *and use substitution* $t = \cos\varphi$ *show that numbers*

$$t_0 = \cos\frac{\alpha}{3}, \quad t_1 = \cos\frac{\alpha+2\pi}{3}, \quad t_2 = \cos\frac{\alpha+4\pi}{3}$$

give full set of all real roots of equation $4t^3 - 3t = d$ *if* $|d| \le 1$. *Find how multiplicity of a root depend from the value of* d.

2. Consider equation $4t^3 - 3t = d$ & $|d| > 1$ *and prove that*

$$t_0 = \frac{\sqrt[3]{d+\sqrt{d^2-1}} - \sqrt[3]{d-\sqrt{d^2-1}}}{2}$$ *is a unique real root in considering case*

(Hint. Prove that $|t_0| \ge 1$*).*

3. Consider equation $4t^3 + 3t = d$ *and prove that*

$$t_0 = \frac{\sqrt[3]{\sqrt{d^2+1} + d} - \sqrt[3]{\sqrt{d^2+1} - d}}{2}$$ *is a unique root of this equation.*

4. Substitute $y = 2\cdot\sqrt{\frac{|b|}{3}}\cdot t$ *into equation* $y^3 + by + c = d$ *and prove that in the cases of different sign of* b *(we suppose* $b\ne 0$ & $c\ne 0$*) we get the equation equivalent to the given in either of considered above forms.*

5. Formulate the results derived in the 1,2,3 for equation $y^3 + by + c = 0$ *by using substitution* $t = \dfrac{y}{2\cdot\sqrt{\frac{|b|}{3}}}$.

Which role in classifications of possible cases plays expression

$D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$. *Find as you can the explicit formulas for solution equation* $y^3 + by + c = 0$ *in* $\mathbb{R}$ *relatively* D.

(*Hint. Consider* $D \le 0$ & $D > 0$).

(v). By considering complex domain appears new possibilities with new questions.

We write the cubic equation $x^3 - r\cdot x^2 + p\cdot x - q = 0$.

1. Prove that this cubic equation always has 3 solution in the set of complex numbers (the root taken account much times as multiplicity it has)
2. If we write the discriminant Δ of quadratic equation $x^2+px+q=0$, by using roots of quadratic equation, then $\Delta=p^2-4q=(x_1+x_2)^2 - 4x_1x_2=(x_1-x_2)^2$.

   Similarly, for cubic equation

   $\Delta=(x_1-x_2)^2\cdot(x_1-x_3)^2\cdot(x_2-x_3)^2$ call discriminant.

   Prove following statements:

a) $\Delta>0$, then all of three roots are distinct
b) $\Delta=0$, then among roots of equation there are two which are equal
c) $\Delta<0$, then one root is real and two others are complex conjugate.
d) how Δ is connected with D.

However for distinction of situation with the root considering of discriminant is insufficiently, since Δ doesn't distinct cases when one of roots is of multiplicity 2 and another is simple from the case when one root of multiplicity 3. For distinction usually use expression

$$\Delta_1 = (x_1-x_2)^2 + (x_2-x_3)^2 + (x_1-x_3)^2.$$

Prove that if $\Delta=0$ and

a) $\Delta_1=0$, then there is one root of multiplicity 3.

b) $\Delta_1\neq0$, then there are 2 roots: one of multiplicity 2, another is simple.

And $\Delta\ge0$ is necessary condition in order to all roots be real.

Prove: $\Delta_1=2\cdot(r^2-3p)^2$; $\Delta=-4r^3q + r^2p^2 + 18\cdot rpq - 4p^3 - 27q^2$

Prove theorem. For equation has all root be real and positive it's necessary and sufficiently that

$$\begin{cases} r\ge0,\ p\ge0,\ q\ge0 \\ \Delta \ge 0 \end{cases}$$

Try to prove results of this item without using complex using and analysis.
We are interesting in connections geometry of triangle with cubic three terms (see article "Orthogonal elements in triangle).
Here we show the connection by following general statements:
1. Prove that by segments with lengths x_1, x_2, x_3 we can build a triangle if and only if

$$(x_1+x_2-x_3)\cdot(x_1-x_2+x_3)\cdot(-x_1+x_2+x_3) > 0$$

2. By segments with the lengths which are roots of equation

$$x^3 + r\cdot x^2 + p\cdot x + q = 0$$

we can build a non-degenerated triangle if and only if coefficients r,p,q of this equation when $\Delta \ge 0$, $r>0$, $p>0$, $q<0$ satisfy inequality

$$r^3 - 4\cdot rp + 8\cdot q > 0$$

This theme has unexpected continuation and applications. To be continued.

NOTES ON THE MARGIN

Let $P(x) = x^k + a_1\cdot x^{k-1} + \ldots + a_k$ and $a_1, a_2, \ldots, a_k$ be an integer.

Then $P(x)=0$ either has no rational roots or has integer roots with each root must be a divisor of a_k.

Proof:

Let $x = \frac{m}{n} \in \mathbb{Q}$ and m & n be a mutually coprime numbers. Then $P(\frac{m}{n}) = 0 \iff$

$$\iff m^k + a_1\cdot m^{k-1}\cdot n + \ldots + a_k\cdot n^k = 0 \iff$$

$$\iff m^k = n\cdot(-a_1\cdot m^{k-1} + \ldots + a_k\cdot n^{k-1})$$

$\iff m^k \mathrel{\vdots} n$. Since n and m are mutually coprime then $n=1$.

So, if the root is a rational number, then it is an integer. If $x=m$ is integer root then

$$m^k + a_1\cdot m^{k-1} + \ldots + a_{k-1}\cdot m + a_k = 0 \iff$$

$$\iff m\cdot(-a_{k-1} - \ldots - m^{k-1}) = a_k \iff a_k \mathrel{\vdots} m.$$

Hence, integer root must be a divisor of free term. If neither integer divisor of free term is a root of equation $P(x)=0$ then it has no rational roots.

Example: $x^3 - 2x^2 - 5x + 6 = 0$. The set of divisors of 6 is $\{\pm1, \pm2, \pm3, \pm6\}$ Among them $1, -2, 3$ are roots. Since given equation can have no more than three roots we deduce $x_1=1$, $x_2=-2$, $x_3=3$ are roots of given equation.

GOOD BOUNDS

Alt Arkady
"Blikh" school
Ramat-Gan,

Appearance of this notes is obligated to an innocent question as it seems for outside vision:

« *To find pretty good bounds for the sum* $S_n = \sum_{k=1}^{n} \operatorname{arctg}\sqrt{k}$, *i.e. locate* S_n *into interval* (m_n, M_n), *where* m_n *and* M_n *some values depending from* n *and more convenient for evaluaiton than* S_n ».

I've been involved in this problem and look what came out.

I'll try to restore the way I've thought:

First of all I considered this sum as integral sum corresponded to the partitions of segment $[0,n]$ with unit step.

(Notice that function $f(x) = \operatorname{arctg}\sqrt{x}$ is convex on $(0,\infty)$. Actually,

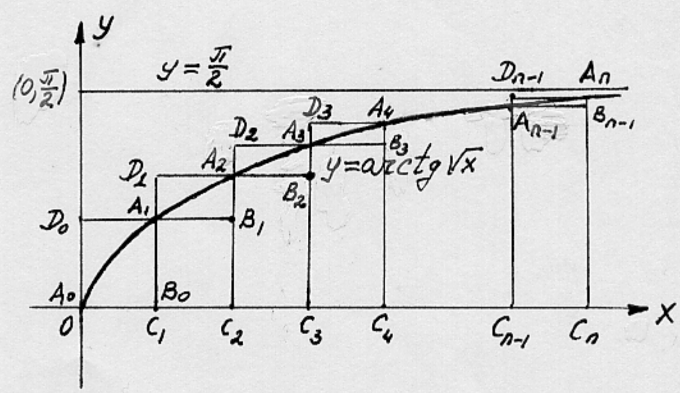
$$f'(x) = \frac{1}{2\sqrt{x}\cdot(1+x)} \quad \& \quad f''(x) = -\frac{(\sqrt{x}\cdot(1+x))'}{2(\sqrt{x}\cdot(1+x))^2} < 0.).$$

The area of curvilinear trapezoid bounded by X-axis & by curve $y = \operatorname{arctg}\sqrt{x}$ onto segment $[0,n]$ equal to $\int_0^n \operatorname{arctg}\sqrt{x}\,dx$.

Since c_i has coordinates $(i,0)$, then

$$\sum_{i=0}^{n-1} S_{A_iD_iA_{i+1}C_{i+1}} = \sum_{i=0}^{n-1} \operatorname{arctg}\frac{1}{\sqrt{i+1}} = S_n,$$

$$\sum_{i=1}^{n-1} S_{C_iA_iB_iC_{i+1}} = \sum_{i=1}^{n-1} \operatorname{arctg}\frac{1}{\sqrt{i}} = S_{n-1},$$



Figure

it follows that

$$S_{n-1} < \int_0^n \operatorname{arctg}\sqrt{x}\,dx < S_n.$$

Calculate $\int_0^n \operatorname{arctg}\sqrt{x}\,dx$. We'll do it by the integration by parts

$$\int_0^n \operatorname{arctg}\sqrt{x}\,dx = \left[\begin{matrix} u'=1 & u = x \\ v = \operatorname{arctg}\sqrt{x}; & v' = \frac{1}{2\sqrt{x}\cdot(1+x)} \end{matrix}\right] = (x\cdot\operatorname{arctg}\sqrt{x})\Big|_0^n - \int_0^n \frac{x\,dx}{2\sqrt{x}(1+x)}$$

$$= n\cdot\operatorname{arctg}\sqrt{n} - \int_0^n \frac{x+1-1}{2\sqrt{x}\cdot(1+x)}dx = n\cdot\operatorname{arctg}\sqrt{n} - \int_0^n \frac{dx}{2\sqrt{x}} + \int_0^n \frac{dx}{2\sqrt{x}(1+x)} =$$

$$= n\cdot\operatorname{arctg}\sqrt{n} - \sqrt{n} + \operatorname{arctg}\sqrt{n} = (n+1)\cdot\operatorname{arctg}\sqrt{n} - \sqrt{n}.$$

Denote $(x+1)\cdot \operatorname{arctg}\sqrt{x} - \sqrt{x}$ by $F(x)$. $F(x)$ is primitive for $f(x) = \operatorname{arctg}\sqrt{x}$, since $F'(x) = f(x)$.

Since $S_{n-1} < F(n) < S_n$, then $S_n \in (m_n, M_n)$, where $m_n = F(n)$ & $M_n = F(n+1)$.

Look at behavior of length of an interval into which located the sum S_n with $n\to\infty$:

$$M_n - m_n = (n+2)\cdot \operatorname{arctg}\sqrt{n+1} - (n+1)\cdot \operatorname{arctg}\sqrt{n} - (\sqrt{n+1} - \sqrt{n}).$$

Since $\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}$ for any $x\neq 0$, then

$$F(n+1) - F(n) = \frac{\pi}{2} + (n+1)\cdot \operatorname{arctg}\frac{1}{\sqrt{n}} - (n+2)\cdot \operatorname{arctg}\frac{1}{\sqrt{n+1}} - (\sqrt{n+1} - \sqrt{n}).$$

From inequality $\sin x < x < \operatorname{tg} x$, for $0 < x < \frac{\pi}{2}$ assuming $x = \operatorname{arctg} t$, $t>0$ we obtain inequality

$$\frac{t}{\sqrt{1+t^2}} < \operatorname{arctg} t < t$$

Hence

$$\frac{1}{\sqrt{x+1}} < \operatorname{arctg}\frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x}}.$$

It follows that

$$\sqrt{n+1} < (n+1)\cdot \operatorname{arctg}\frac{1}{\sqrt{n}} < \frac{n+1}{\sqrt{n}} \quad \text{and} \quad \sqrt{n+2} < (n+2)\cdot \operatorname{arctg}\frac{1}{\sqrt{n+1}} < \frac{n+2}{\sqrt{n+1}}$$

Subtracting second inequality from first inequality we get:

$$\sqrt{n+1} - \frac{n+2}{\sqrt{n+1}} < (n+1)\cdot \operatorname{arctg}\frac{1}{\sqrt{n}} - \frac{n+1}{\sqrt{n}} < \frac{n+1}{\sqrt{n}} - \sqrt{n+2} \iff$$

$$-\frac{1}{\sqrt{n+1}} < (n+1)\cdot \operatorname{arctg}\frac{1}{\sqrt{n}} - (n+2)\cdot \operatorname{arctg}\frac{1}{\sqrt{n+1}} < \frac{n+1-\sqrt{n\cdot(n+2)}}{\sqrt{n}} = \frac{1}{\sqrt{n}\cdot(n+1+\sqrt{n(n+2)})}$$

Since $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$ and $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}(n+1+\sqrt{n(n+2)})} = 0$, then

$$\lim_{n\to\infty}\left((n+1)\cdot \operatorname{arctg}\frac{1}{\sqrt{n}} - (n+2)\cdot \operatorname{arctg}\frac{1}{\sqrt{n+1}}\right) = 0.$$

Moreover,

$$\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0.$$

Thus, $\lim\limits_{n\to\infty}(M_n - m_n) = \frac{\pi}{2}$.

Here we have to underline that $M_n - m_n$ monotone increasing approach $\frac{\pi}{2}$, the proof by Lagrange´s mean value theorem (*).

Actually,

$$M_n - m_n = F(n+1) - F(n) = F'(x_n) = f(x_n) = \operatorname{arctg} \sqrt{x_n}, \text{ when } x_n \in (n,n+1)$$

Hence,

$$M_{n+1}-m_{n+1} = \operatorname{arctg} \sqrt{x_{n+1}} > \operatorname{arctg} \sqrt{n+1} > \operatorname{arctg} \sqrt{x_n} = M_n-m_n.$$

When $n = 1$:

$$M_1-m_1 = F(2)-F(1) = 3\cdot\operatorname{arctg} \sqrt{2} - \sqrt{2} - 2\cdot\operatorname{arctg} 1 + 1 = 3\cdot\operatorname{arctg} \sqrt{2} - \frac{\pi}{2} - \sqrt{2} + 1$$

(*Prove straightly and no using calculator that*

$$3\cdot\operatorname{arctg} \sqrt{2} - \frac{\pi}{2} - \sqrt{2} + 1 < \frac{\pi}{2}).$$

Hence:

$$3\cdot\operatorname{arctg} \sqrt{2} - \frac{\pi}{2} - \sqrt{2} + 1 < M_n-m_n < \frac{\pi}{2}$$

So, the difference between upper and lower bound for S_n increasing approach $\frac{\pi}{2}$. Can we think that we succeed ? So far we have no better results, we can. Anyway I wasn´t satisfied either length of interval $M_n - m_n$ and too cumbersome reasoning whereby we came to result.

Therefore I chose another way, hinted by identity

$$\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}$$

Then

$$S_n = n\cdot\frac{\pi}{2} - \sum_{k=1}^{n} \operatorname{arctg} \frac{1}{\sqrt{k}}.$$

Denote the sum $\sum_{k=1}^{n} \operatorname{arctg} \frac{1}{\sqrt{k}}$ by T_n

and its in what follows will play most important role and as turn out further it will be a motive to turn the reflection on unexpected way.
Now consider the sum T_n.

From inequality

$$\frac{1}{\sqrt{k+1}} < \operatorname{arctg} \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k}} \quad \text{we get} \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}} < T_n < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

Hence the question is to find upper & lower bounds for $\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$.

Here I´d remembered a problem that I´ve solved before.
This problem was offered on one of the Holland olympiads:

we´d prove inequality:

$$1 < 2\cdot\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2$$

We are going to prove it.

$$1 < 2\cdot\sqrt{n} - \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2 \quad\Leftrightarrow\quad 2\cdot\sqrt{n} - 2 < \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\cdot\sqrt{n} - 1.$$

Prove right-hand side inequality in the last double inequality by complete mathematic induction method:

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\cdot\sqrt{n} + \frac{1}{\sqrt{n+1}} - 1.$$ Notice that:

$$\frac{1}{\sqrt{n+1}} < 2\cdot(\sqrt{n+1} - \sqrt{n}) \Leftrightarrow \frac{1}{\sqrt{n+1}} < \frac{2}{\sqrt{n+1} + \sqrt{n}} \Leftrightarrow \sqrt{n+1} + \sqrt{n} < 2\cdot\sqrt{n+1} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{n} < \sqrt{n+1}.$$

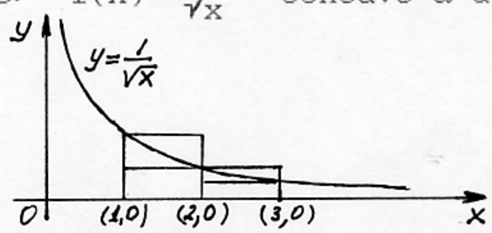
Hence, substituting $\frac{1}{\sqrt{n+1}}$ on $2\cdot(\sqrt{n+1} - \sqrt{n})$

we obtain needed upper bound: $2\cdot\sqrt{n+1} - 1.$

The lower bound we cannot find by the same way. But one special way can be used here, I mean an integral sum:

$$\left(\left(\frac{1}{\sqrt{x}}\right)' = -\frac{1}{2x\sqrt{x}},\ (x^{-1/2})'' = \frac{3}{4x^2\sqrt{x}} > 0,\ \text{i. e.}\quad f(x) = \frac{1}{\sqrt{x}}\quad \text{concave \& decrease}\right).$$

$$\int_1^n \frac{1}{\sqrt{x}}\,dx < \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$



Hence, since

$$\int_1^n \frac{dx}{\sqrt{x}} = (2\cdot\sqrt{x})\Big|_1^n = 2\cdot\sqrt{n} - 2,$$ we get needed inequality.

However this way of proof didn't satisfy me. I've decided to think a little and try to find the elementary proof of that inequality using minimum of means. An idea came to me at the same moment as soon I deliberately limited myself in means, and I got stronger inequality.

Actually,

$$2\cdot(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2\cdot(\sqrt{k} - \sqrt{k-1}) \Leftrightarrow \frac{2}{\sqrt{k-1} + \sqrt{k}} < \frac{1}{\sqrt{k}} < \frac{2}{\sqrt{k} + \sqrt{k+1}},$$

that's obviously true for $k > 1$.

Hence

$$\sum_{k=2}^{n}\frac{1}{\sqrt{k}} < 2\cdot\sqrt{k} - 2 \Leftrightarrow \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\cdot\sqrt{n} - 1 \ \&\ \sum_{k=1}^{n}\frac{1}{\sqrt{k}} > 2\cdot(\sqrt{n+1} - 1) > 2\cdot\sqrt{n} - 2$$

Thus

$$2\cdot\sqrt{n+1} - 2 < \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\cdot\sqrt{n+1} - 1.$$

Go back to initial problem, i.e. to bounds for sum $T_n = \sum_{k=1}^{n}\operatorname{arctg}\frac{1}{\sqrt{k}}$.

We have

$$-1 + \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < T_n < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \iff 2\cdot\sqrt{n+2} - 3 < T_n < 2\cdot\sqrt{n} - 1$$

Hence,

$$m_n = \frac{n\pi}{2} - 2\cdot\sqrt{n} + 1 < S_n < \frac{n\pi}{2} - 2\cdot\sqrt{n+2} + 3 = M_n$$

The length of interval located the sum is equal to

$$2 - 2\cdot(\sqrt{n+2} - \sqrt{n}) = 2 - \frac{4}{\sqrt{n+2} + \sqrt{n}}.$$

So,

$$M_n - m_n = 2 - \frac{4}{\sqrt{n+2} + \sqrt{n}} \quad \& \quad \lim_{n\to\infty} (M_n - m_n) = 2,$$

Even the last estimation for S_n has got by very simple way and that estimation not much worse then integral ($\frac{\pi}{2} \approx 1.5$), however both of them didn´t satisfy me. Under last circumstance I took this problem seriously.

I dropped it and decided to look back and ask to myself follows question 1) What do I want ? ; 2) What does it look like ?
At the moment I answered to first question: I want good asymptotic, i.e. find such function $\varphi(n)$, that $\lim_{n\to\infty} (S_n - \varphi(n)) = 0$, and which more convenient for straightforward calculation then S_n.

As regard second question, the work we have done doesn´t go for nothing, since I remembered that before I´ve met with $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} - 2\cdot\sqrt{n}$, namely I offered for my pupils such problem, to be more precisely, the following series of problems.

Consider two sequences:

$$a_n := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\cdot\sqrt{n+1} \quad \& \quad b_n := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\cdot\sqrt{n}$$

1. Prove a_n is monotone increasing.
2. Prove b_n is monotone decreasing.
3. Prove $b_n - a_n \to 0$ when $n\to\infty$
4. Sequences a_n & b_n approach the same number (denote it by $C_{1/2}$).

Similarly questions for sequences:

$$a_n := \sum_{k=1}^{n} \frac{1}{k} - \ln n \quad \& \quad b_n := \sum_{k=1}^{n} \frac{1}{k} - \ln (n+1).$$

And in this case both of sequences have a common limit C_1, which was called *Euler´s constant* and has special notation $\nu \approx 0.5772156649\ldots$

Obviously, in either cases we have for $\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ asymptotic $2\cdot\sqrt{n} + C_{1/2}$, and for $\sum_{k=1}^{n} \frac{1}{k}$ asymptotic $\ln n + C_1$.

I cannot be silent and I have to pay attention on follows complect of problems on asymptotic for $\sum_{k=1}^{n} \frac{1}{k}$.

Consider sum $P_n = \sum_{k=1}^{n} \frac{1}{k}\cdot(-1)^{k+1}$, and denote $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then:

Problem 1. Prove $P_{2n} = H_{2n} - H_n$.

Problem 2. Prove $\lim_{n\to\infty} P_{2n} = \ln 2$.

Problem 3. Prove $\lim_{n\to\infty} P_n = \ln 2$.

Give back to associations, we´ll be going on, setting before ourself the problem of searching asymptotic for T_n. Considered above problems put us on to an idea that in this case we can do the same by offered scheme, considering consequently

$$a_n := T_n - F(n+1) \quad \& \quad b_n := T_n - F(n),$$

with $F(x)$ - primitive of function $f(x) = \operatorname{arctg}\frac{1}{\sqrt{x}}$

as I did, but I had defined before a primitive. When I finished the reducing I ask myself: what does compel me to tie reasoning to concrete function $f(x)$ if everything point to common character of reasoning ?
Transition to general problem as it happening frequently, visibly reduced technical work and naked essence of problem. So, let

$T_n(f) := \sum_{k=1}^{n} f(k)$, with $\lim_{x\to\infty} f(x) = 0$ and

$f(x)$ monotone decreasing, differntiable, positive function.

$F(x)$ is the primitive for $f(x)$ onto $[1,\infty)$.
Obviously $F(x)$ is monotone *in*creasing. More accurate definition of $f(x)$ and $F(x)$ will do by necessative.

Consider sequences: $a_n := T_n(f) - F(n+1)$ & $b_n := T_n(f) - F(n)$.

Obviuosly, for any $n \in \mathbb{N}$ $a_n < b_n$.

Prove a_n - monotone increasing & b_n - monotone decreasing functions:

1. $a_{n+1} - a_n = (T_{n+1}(f)-F(n+2)) - (T_n(f)-F(n+1)) =$

$= (T_{n+1}(f)-T_n(f)) - (F(n+2)-F(n+1)) = f(n+1) - F'(d_n) = f(n+1) - f(d_n)$,

where $d_n \in (n+1,n+2)$ by Lagrange´s Mean value theorem (*).

But since

$d_n > n+1$, then $f(d_n) < f(n+1)$, since $f(x)$ is monotone decreasing.

Therefore, $a_{n+1} - a_n > 0$.

2. $b_{n+1} - b_n = (T_{n+1}(f)-F(n+1)) - (T_n(f)-F(n)) =$

$$= (T_{n+1}(f)-T_n(f)) - (F(n+1)-F(n)) = f(n+1) - F'(\bar{d}_n) = f(n+1) - f(\bar{d}_n),$$

where $\bar{d}_n \in (n,n+1)$ by Lagrange´s theorem.

Since $\bar{d}_n < n + 1$, then $f(\bar{d}_n) > f(n+1)$, therefore $b_{n+1} - b_n < 0$.

3. Since $a_1 < a_n < b_n < b_1$, then the both of sequences have limits as monotone and bounded sequences.

4. $b_n-a_n = F(n+1)-F(n) = f(\tilde{d}_n)$, where $\tilde{d}_n \in (n,n+1)$.

Hence, $f(n+1) < b_n - a_n < f(n)$

and it follows, $\lim_{n\to\infty} (b_n-a_n) = 0$.

Thus, the both of sequences converge to the same number which in what follows we´ll call C_f.

So, $a_n < C_f < b_n$ for any $n \in \mathbb{N}$, i.e.

$$T_n-F(n+1) < C_f < T_n-F(n) \iff F(n) + C_f < T_n(f) < F(n+1) + C_f$$

if assume $\varphi(n) := F(n) + C_f$, then occur inequality:

$$\varphi(n) < T_n(f) < \varphi(n+1),$$

here the upper and lower bound can´t be improved, since $\lim_{n\to\infty} (\varphi(n+1)-\varphi(n)) = 0$ and function $\varphi(n)$ is asymptotic for $T_n(f)$, since $\lim_{n\to\infty} (T_n(f)-\varphi(n)) = 0$

For asymptotic $T_n(f)$, where $f(x) = \text{arctg}\frac{1}{\sqrt{x}}$ we have to verify holding of the demands to function $f(x)$ and find its primitive.

1. $f(x)$ is monotone decreasing, since

$\text{arctg}\frac{1}{\sqrt{x}} = \frac{\pi}{2} - \text{arctg}\sqrt{x}$ & $\text{arctg}\sqrt{x}$ is monotone increasing.

2. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \text{arctg}\frac{1}{\sqrt{x}} = 0$.

3. We have the primitive for $\text{arctg}\sqrt{x}$, its equal to $(x+1)\cdot\text{arctg}\sqrt{x} - \sqrt{x}$.

Since $f(x) = \frac{\pi}{2} - \text{arctg}\sqrt{x}$, then

$$F(x) = \frac{\pi}{2}x - (x+1)\cdot\text{arctg}\sqrt{x} + \sqrt{x} + c = c + \frac{\pi}{2}x - (x+1)\cdot(\frac{\pi}{2} - \text{arctg}\frac{1}{\sqrt{x}}) + \sqrt{x}$$

$$= (x+1)\cdot\text{arctg}\frac{1}{\sqrt{x}} + \sqrt{x} - \frac{\pi}{2} + c.$$

Assuming $c = \frac{\pi}{2}$ we obtain $F(x) = (x+1)\cdot\text{arctg}\frac{1}{\sqrt{x}} + \sqrt{x}$.

Then if

$$C_f = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \operatorname{arctg}\frac{1}{\sqrt{k}} - (n+1)\cdot\operatorname{arctg}\frac{1}{\sqrt{n}} - \sqrt{n} \right),$$

then

$$\mu(n) := (n+1)\cdot\operatorname{arctg}\frac{1}{\sqrt{n}} + \sqrt{n} + C_f$$

– is asymptotic for the sum $T_n = \sum_{k=1}^{n} \operatorname{arctg}\frac{1}{\sqrt{k}}$ and holding inequality:

$$\boxed{C_f + (n+1)\cdot\operatorname{arctg}\frac{1}{\sqrt{n}} < \sum_{k=1}^{n} \operatorname{arctg}\frac{1}{\sqrt{k}} < C_f + (n+2)\cdot\operatorname{arctg}\frac{1}{\sqrt{n+1}} + \sqrt{n+1}}$$

The value of C_f what we´ve evaluated on computer equal to
$C_f = -2.1474246867$ when $n=2330$.

Note. All these reasoning are holding for f(x) which is monotone increasing and has horizontal asymptotic.

(*). *Appendix:*

Rollya´s Theorem. Let f(x) is continuous function onto [a,b] and differentiable onto (a,b) & f(a) = f(b), then there exist $c \in (a,b)$ such that

$$f'(c) = 0.$$

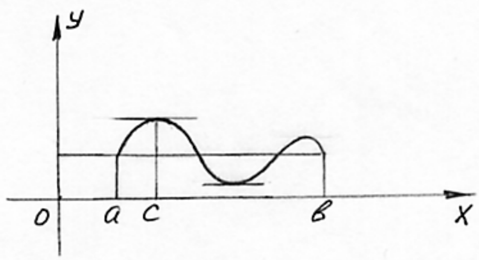
Proof: Denote $m = \min_{x\in[a,b]} f(x) = f(x_*)$ & $M = \max_{x\in[a,b]} f(x) = f(x^*)$,

where x_* & x^* some points of segment [a,b]. It´s possible two cases:

1. Two-elements sets are equal: $\{x_*, x^*\} = \{a,b\}$, then m = M, since

   f(a) = f(b) and it follows, f(x) = const & $f'(x) = 0$ for any $x \in (a,b)$.

2. $\{x_*, x^*\} \neq \{a,b\}$, then at least either x_* or x^* belongs to interval (a,b) Function f(x) into that point, has extremum. If we denote this point by c, we get $f'(c) = 0$ & $c \in (a,b)$. ■



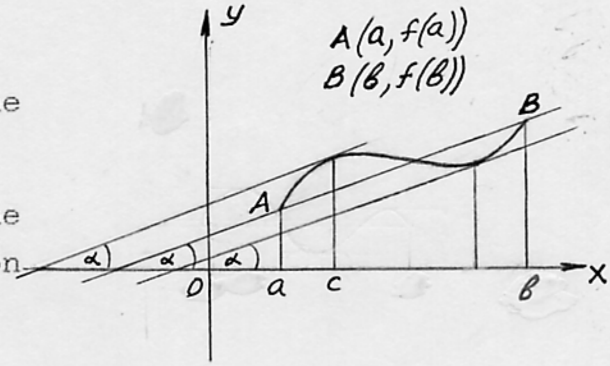
Geometric mean is: if function f(x) satisfy the theorem then we can find at least one point $c \in (a,b)$ such that tangent to the function graph y = f(x) in the point with c-absciss is parallel to the X-axis.

Mean value Theotem. Let f(x) is continuous function onto segment [a,b] and differetiable onto interval (a,b), then there exist $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

(Geometric mean is: we can find the point $c \in (a,b)$, such that the tangent to the function graph with c-absciss is parallel to the chord connected points $(a,f(a))$ and $(b,f(b))$).

Proof:



$y=\dfrac{f(b)-f(a)}{b-a}\cdot(x-a) + f(a)$ – equation of line going through points $A(a,f(a))$ & $B(b,f(b))$, then since the points A & B are laying on the graph of $y = f(x)$ and on line, then function

$$\varphi(x) = f(x) - \frac{f(b)-f(a)}{b-a}\cdot(x-a) - f(a)$$

(as continuous function on $[a,b]$ and differentiable on (a,b)) in the points a & b equal to zero.
Then by Rollya´s theorem there exist $c \in (a,b)$, such that

$$\varphi'(c)=0; \qquad \varphi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Thus

$$\varphi'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}.$$

NOTES ON THE MARGIN

From any equation of the fourth degree $x^4 + a\cdot x^3 + b\cdot x^2 + c\cdot x + d = 0$ by substitution $x = y - \frac{a}{4}$ we get equaiton $y^4 + b'\cdot y^2 + c'\cdot y + d' = 0$.

Therefore, worth while to consider equation of 4-th degree as $x^4+bx^2+cx+d=0$.

Let p be a parameter, then equation can be rewritten

$(x^2+p)^2-2p\cdot x^2-p^2+b\cdot x^2+c\cdot x+d = 0 \iff (x^2+p)^2 = x^2\cdot(2p-b) -c\cdot x^2 + p^2 - d$

In the right-hand side we have quadratic polynomial. By choosing p we can obtain that this quadratic polynomial be an exact square accurate to a sign, i.e. $\pm(k\cdot x + l)^2$. For that sufficiently choose p so that the discriminant of quadratic polynomial be equal to 0, i.e. $c^2 + 4\cdot(d-p^2)\cdot(2p-b) = 0$.

Derived equation is an equation of the third degree relatively p. And an equation of the third degree always has solution. Substitution derived value of p into the equation yields $(x^2+p)^2 = \pm(kx+l)^2$.

If the sign of exact square is minus, then given equation has no real roots. If the sign of exact square is plus then we get two quadratic equations

$$\left[\begin{array}{l} x^2 + p = kx + l \\ x^2 + p = -kx - l \end{array}\right.$$

By solving them we obtain the solutions of given equation of the 4-th degree.

"DELTA"'S SCHOOL

VARIATION ON INEQUALITY THEME

Alt Arkady
"Blikh" school
Ramat-Gan,

The reader who doesn't know some elements of analysis (limits, continuity, derivatives) can pass those parts without any damage for understanding the rest of article (these parts have another print type).

Variation 1.

Consider inequality which holds for any real numbers x & y:

$$(x-y)^2 \geq 0 \qquad (1)$$

Its obviously equivalent inequality:

$$\frac{x^2 + y^2}{2} \geq x \cdot y \qquad (2)$$

Where equality is possible when x=y. If a and b are positive numbers, then setting $x=\sqrt{a}$ & $y=\sqrt{b}$ we get Cauchy inequality (particular case):

$$\frac{a + b}{2} \geq \sqrt{ab} \qquad (3)$$

Equality is possible when a=b only. But inequality (3) we could get straightforward from the trivial inequality $(\sqrt{a} - \sqrt{b})^2 \geq 0$.
Consider some problems on inequality proving, for which solution require special and unordinary ways of using inequalities in form (2) or (3).

Example 1. Prove inequality $a^2 + b^2 + c^2 \geq ab + bc + ac$, where a, b, c – non-negative numbers.

Solution:

$$a^2 + b^2 + c^2 = \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} \geq ab + bc + ac.$$ Equality occur when a=b=c.

Example 2. Prove inequality $a^4 + b^4 + c^4 \geq abc \cdot (a+b+c)$ for non-negative numbers a, b, c.

Solution:

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 = \frac{a^2b^2+b^2c^2}{2} + \frac{b^2c^2+c^2a^2}{2} + \frac{c^2a^2+a^2b^2}{2} \geq$$
$$\geq b^2ac + c^2ab + a^2bc = abc \cdot (a+b+c).$$

Example 3. For $x, y, z \geq 0$ prove inequality: $xy + yz + zx \geq \sqrt{3xyz \cdot (x+y+z)}$.

Solution:

$$(xy+yz+zx)^2 = x^2y^2+y^2z^2+z^2x^2+2xy \cdot yz+2yz \cdot zx+2xy \cdot zx = (x^2y^2+y^2z^2+z^2x^2) +$$
$$+ 2xyz \cdot (x+y+z) \geq xyz \cdot (x+y+z) + 2xyz \cdot (x+y+z) = 3xyz \cdot (x+y+z).$$

Exercise 1. Prove for positive numbers a, b, c inequality

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Exercise 2. Prove for non-negative numbers a, b, c follows inequalities

a) $a^2\cdot(1+b^2) + b^2\cdot(1+c^2) + c^2\cdot(1+a^2) \ge 6\cdot abc$

б) $6\cdot abc \le ab\cdot(a+b) + bc\cdot(b+c) + ca\cdot(c+a) \le 2\cdot(a^3+b^3+c^3)$

Exercise 3. Find out what's greater:

a) $2+\sqrt{3}$ or $\sqrt[4]{192}$

b) $\sqrt[3]{4} - \sqrt[3]{10} + \sqrt[3]{25}$ or $\sqrt[3]{6} - \sqrt[3]{9} + \sqrt[3]{15}$

Exercise 4.

a) Prove inequality for non-negative a & b:

$$2\cdot\sqrt{a^2+ab+b^2} \ge \sqrt{3}\cdot(a+b)$$

б) Prove inequality for non-negative x, y, z

$$\sqrt{x^2+xy+y^2} + \sqrt{y^2+yz+z^2} + \sqrt{z^2+zx+x^2} \ge \sqrt{3}\cdot(x+y+z)$$

в) Prove inequality for positive x, y, z

$$\frac{x+y+z}{3\sqrt{3}} \ge \frac{xy+yz+xz}{\sqrt{x^2+xy+y^2}+\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}}$$

Follows series of problems which are especially that for their solution unsufficient inequality (2) or (3). We can get desired result by combine Cauchy inequality with another very important inequality which we can consider as base inequality. Consider two ordered pair of numbers (a, b) & (c, d). (Ordered pair, it means that each of two numbers has its place in the pair. In this meaning, the pairs $(a,b)\ne(b,a)$ and pairs (a, b) & (c, d) are equals if and only if $a=c$ & $b=d$.)

Definition. In what follows we shall say (a, b) & (c, d) are concordant in order, if simultaneously $a\le b$ & $c\le d$ or $a\ge b$ & $c\ge d$. Otherwise, we saying that pairs are not concordant.
Follows statement is trivial. Pairs (a, b) & (c, d) are concordant in order if and only if $(a-b)\cdot(c-d)\ge 0$.
So, let pairs (a, b) & (c, d) are concordant in order, then:

$$ac + bd \ge ad + bc \qquad (4)$$

For instance:

1. Pairs (x^2, y^2) & (x^3, y^3) are concordant in order and it means that holds inequality $x^5 + y^5 \ge x^2y^3 + y^2x^3$.

2. Pairs $(\sin^3 x, \cos^3 x)$ & $\left(\frac{1}{\cos x}, \frac{1}{\sin x}\right)$ are concordant for $(0, \frac{\pi}{2})$.

Hence, $\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x} \ge 1$.

3. Let functions f(x) & g(x) are monotone increasing (decreasing) on some range D, then pairs (f(x), f(y)) & (g(x), g(y)) are concordant in order for any x and y from this domain and holds inequality:

$$f(x)\cdot g(x) + f(y)\cdot g(y) \geq f(x)\cdot g(y) + f(y)\cdot g(x), \quad x,y\in\mathbb{R} \qquad (5)$$

4. Let $x,y>0$, then pairs (x,y) & $(\frac{1}{y}, \frac{1}{x})$ are concordant in order. More generally, if f & g have different monotone character (one is decreasing, another is increasing), then pairs $(f(x),f(y))$ & $(g(y),g(x))$ are concordant in order and it means:

$$f(x)\cdot g(y) + f(y)\cdot g(x) \geq f(x)\cdot g(x) + f(y)\cdot g(y) \qquad (6)$$

Now go back to problems.

Example 4. Prove inequality $a^5 + b^5 + c^5 \geq abc\cdot(ab+bc+ca)$, where $a,b,c\geq 0$.

Solution: Since for any non-negative x and y pairs (x,y) & (x^4,y^4) are concordant in order, then $x^5 + y^5 \geq x^4y + xy^4$. Hence,

$$a^5 + b^5 + c^5 = \frac{a^5+b^5}{2} + \frac{b^5+c^5}{2} + \frac{c^5+a^5}{2} \geq \frac{a^4b+ab^4}{2} + \frac{b^4c+bc^4}{2} + \frac{c^4a+ca^4}{2} =$$

$$\frac{a^4b+bc^4}{2} + \frac{b^4c+ca^4}{2} + \frac{c^4a+ab^4}{2} \geq a^2bc^2 + b^2ca^2 + c^2ab^2 = abc\cdot(a+b+c).$$

Example 5. For $a,b,c>0$ prove inequality

$$a + b + c \leq \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b} \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}$$

Solution:

1. $\frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b} \geq \frac{2ab}{2c} + \frac{2bc}{2a} + \frac{2ac}{2b} = \frac{1}{2}\left(\frac{ab}{c} + \frac{bc}{a}\right) + \frac{1}{2}\left(\frac{bc}{a} + \frac{ac}{b}\right) +$

$+\frac{1}{2}\left(\frac{ab}{c} + \frac{ac}{b}\right) \geq b + c + a.$

2. $\frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab} = \frac{a^4+b^4+c^4}{abc}$. But pairs (x,y) & (x^3,y^3) are concordant in order for any x & y, and it means $x^4 + y^4 \geq x^3y + xy^3$.

Therefore,

$$a^4 + b^4 + c^4 = \frac{a^4+b^4}{2} + \frac{b^4+c^4}{2} + \frac{a^4+c^4}{2} \geq \frac{1}{2}(a^3b+ab^3) + \frac{1}{2}(b^3c+bc^3) + \frac{1}{2}(c^3a+ac^3) =$$

$$= \frac{1}{2}ab(a^2+b^2) + \frac{1}{2}bc(b^2+c^2) + \frac{1}{2}ac(a^2+c^2).$$

It follows that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b}.$$

Sometimes difficult to get concordant pairs, however, if we can get them then it could be sufficient for proving some difficult inequalities. for instance, inequality from next example (IMO IV - International Mathematics Olympiad).

Example 6. Prove for $a,b,c\geq 0$ inequality:

$$(a+b+c)^3 - 4\cdot(a+b+c)\cdot(ab+bc+ac) + 9\cdot abc \geq 0$$

Solution:

Remove the bracket, perform operations and group similar terms, we get inequality:

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + ab^2 + bc^2 + c^2a + ac^2.$$

Since this inequality is not changing by permutations of numbers a, b, c, then we can define that $a\geq b\geq c$. So,

$$a^3 + b^3 + c^3 + 3abc = a\cdot(a^2+bc) + b\cdot(b^2+ac) + c\cdot(c^2+ab).$$

Pairs (a, b) & (a^2+bc, b^2+ac) are concordant, since

$$a^2+bc - (b^2+ac)=(a-b)\cdot(a+b) - c\cdot(a-b) = (a-b)\cdot(a+b-c) \geq (a-b)\cdot(b-c) \geq 0.$$

Therefore,

$$a(a^2+bc) + b(b^2+ac) \geq a(b^2+ac) + b(a^2+bc).$$

Thus

$$a^3 + b^3 + c^3 + 3abc \geq a\cdot(b^2+ac) + b\cdot(a^2+bc) + c\cdot(c^2+ab) = ab^2 + a^2c + ba^2$$
$$+ b^2c + c^3 + abc = (a^2b+ab^2) + a^2c + bc^2 + c^3 + abc.$$

To complete the proof we have to show that

$$c^3 + abc \geq ac^2 + bc^2 \iff c\cdot(c^2+ab-ac-bc) \geq 0 \iff c\cdot(c-a)\cdot(c-b) \geq 0. \blacksquare$$

In this example we have not used the Cauchy´s inequality. However, this example in this place appeared by some purpose. Namely, we´re going to pay attention by some period of time on using concordant pairs, moreover, the Cauchy inequality $x^2+y^2\geq 2xy$ is equivalent to trivial fact - concordance of two pairs (x, y) & (x, y). But as you have seen the corollaries from this trivial fact are not so trivial.

Rewrite that inequality in another form, supposing $x,y > 0$,

$$\frac{x^2}{y} \geq 2\cdot x - y. \qquad (7), \quad \text{equality when } x=y.$$

We just rewrote the trivial inequality in another form and the essence has not changed. But with help of new form the inequality (1) acquire new possibility for using.

Example 7. (IMO XXIV). Prove inequality:

$$x^3z + y^3x + z^3y \geq x^2yz + xy^2z + xyz^2 \quad , \quad x,y,z\geq 0$$

Solution:

Since cases $x=0$ or $y=0$ or $z=0$ are obvious then we´ll suppose that $x,y,z>0$. Therefore, for positive x,y,z by straightforward division by xyz we obtain inequality: $\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z.$

But using inequality (7) gives opportunity to prove it at one line

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq (2x-y) + (2y-z) + (2z-x) = x + y + z.$$

Where equality occur only when $x=y=z$.
By using others concordant pairs we can get other base inequality from which trivial corollaries are not trivial inequalities.
For instance:

1. Inequality $x^4 + y^4 \geq x^3y + xy^3$ is corresponded to concordant pairs (x,y) & (x^3,y^3) and when $x,y>0$ it could be rewritten: $\frac{x^4}{y} \geq x^3 - y^3 + xy^2$.

Hence, right here for $x,y,z>0$ it follows that $\frac{x^4}{y}+\frac{y^4}{z}+\frac{z^4}{x}\geq xy^2+yz^2+zx^2$,

which could be written so: $\frac{x^3}{y^2z}+\frac{y^3}{z^2x}+\frac{z^3}{x^2y} \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ or

like this: $x^5z + y^5x + z^5y \geq xy^2z^3 + x^3yz^2 + x^2y^3z$ for $x,y,z\geq 0$.

(Try to solve those inequalities by other way and you'll understand that it couldn't be so easy !)

2. Inequality $x^5 + y^5 \geq x^3y^2 + y^3x^2$ is corresponded to concordant pairs (x^2,y^2) & (x^3,y^3) for $x,y\geq 0$. In case $x,y>0$ it follows that

$\frac{x^5}{y^2} \geq x^3-y^3+yx^2$, or by straightforward division by x^2y^2 : $\frac{x^3}{y^2} + \frac{y^3}{x^2} \geq x + y$.

3. Inequality $x^5 + y^5 \geq x^4y + xy^4$ is corresponded to concordant pairs (x,y) & (x^4,y^4) for $x,y\geq 0$ and in supposition $x,y>0$ it could be rewritten so:

$\frac{x^5}{y} \leq x^4-y^4+xy^3$ or like this: $\frac{x^4}{y} + \frac{y^4}{x} \geq x^3 + y^3$.

Exercise 5. Prove inequality for $a,b,c \geq 0$

$$a^3 + b^3 + c^3 \geq a^2\sqrt{bc} + b^2\sqrt{ca} + c^2\sqrt{ab}$$

Exercise 6. Prove inequality for $x,y,z>0$

$$\text{a)}\ \frac{x^4}{y^3z} + \frac{y^4}{z^3x} + \frac{z^4}{x^3y} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$\text{б)}\ x^4\cdot\left(\frac{1}{y}+\frac{1}{z}\right) + y^4\cdot\left(\frac{1}{x}+\frac{1}{z}\right) + z^4\cdot\left(\frac{1}{y}+\frac{1}{x}\right) \geq 2\cdot(x^3+y^3+z^3)$$

Exercise 7. Consider concordant pairs (x^n,y^n) & (x^m,y^m) and try to generalize inequalities appeared above.

Exercise 8. Prove for any $a,b,c>0$ inequalities:

$$\text{a)}\ \frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

b) $\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{abc}$.

Variation 2.

Actually, this is a continuation of theme considered above, but in general form. We start as before, from Cauchy inequality.

Definition 1. For any non-negative $a_1,a_2,\ldots,a_n$ the values $\frac{a_1+a_2+\ldots+a_n}{n}$ & $\sqrt[n]{a_1a_2\ldots a_n}$ are called arithmetic average and geometric average respectively

If $a_1,\ldots,a_n>0$, then the value $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}$ is called harmonic average.

The value $G_p=\left(\frac{a_1^p+\ldots+a_2^p}{n}\right)^{1/p}$ for any $p\neq 0$ called degree average of p-degree.

Prove that $\lim\limits_{p\to 0}G_p=\sqrt[n]{a_1a_2\ldots a_n}$ and therefore the notation G_0 is suitable for geometric average since $G_0=\lim\limits_{p\to 0}G_p$.

Proof:

Denote $\frac{a_1^p+\ldots+a_n^p}{n}-1$ by S_p. Obviously, $\lim\limits_{p\to 0}S_p=0$.

Find $\lim\limits_{p\to 0}\frac{S_p}{p}$. We have $\frac{S_p}{p}=\frac{1}{n}\cdot\left(\frac{a_1^p-1}{p}+\ldots+\frac{a_n^p-1}{p}\right)$. Since $\lim\limits_{t\to 0}\frac{a^t-1}{t}=\ln a$,

for any $a>0$ & $a\neq 1$, then $\lim\limits_{p\to 0}\frac{S_p}{p}=\frac{\ln a_1+\ln a_2+\ldots+\ln a_n}{n}=\ln\sqrt[n]{a_1a_2\ldots a_n}$

Then $\lim\limits_{p\to 0}G_p=\lim\limits_{p\to 0}\left[(1+S_p)^{1/S_p}\right]^{S_p/p}=\left(\lim\limits_{s\to 0}(1+\frac{1}{s})^s\right)^{\lim\limits_{p\to 0}\frac{S_p}{p}}=e^{\ln\sqrt[n]{a_1a_2\ldots a_n}}$

$=\sqrt[n]{a_1a_2\ldots a_n}$. We leave to prove $\lim\limits_{t\to 0}\frac{a^t-1}{t}=\ln a$. Since $\lim\limits_{s\to 0}(1+s)^{1/s}=e$,

then $\lim\limits_{s\to 0}\frac{\ln(1+s)}{s}=1$. By notation $\ln(1+s)=t$, we get $\lim\limits_{s\to 0}t=0$ and

$s=e^t-1$. Hence, $\lim\limits_{s\to 0}\frac{s}{\ln(1+s)}=\lim\limits_{t\to 0}\frac{e^t-1}{t}=1$ and it follows

$\lim\limits_{t\to 0}\frac{a^t-1}{t}=\lim\limits_{t\to 0}\frac{e^{\ln a^t}-1}{t\cdot\ln a}=\ln a\cdot\lim\limits_{t\to 0}\frac{e^{t\cdot\ln a}-1}{t\cdot\ln a}=\ln a\cdot 1=\ln a$ ■

The word "average" present here because for any p:
$\min\{a_1,a_2,\ldots,a_n\}\le G_p\le\max\{a_1,a_2,\ldots,a_n\}$. *Prove it by yourself.*
When n=2 we have follows chain of inequalities:

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$ or as we've noted $G_{-1} \le G_0 \le G_1 \le G_2$.

We are going to transfer those inequalities to cases with more variables. First of all we concentrate on central part of this chain.
In general case inequality $G_0 \le G_1$ is to be said Cauchy inequality and it could be written so:

$$\frac{a_1+a_2+\ldots+a_n}{n} \ge \sqrt[n]{a_1a_2\ldots a_n}$$

Easy to prove inequality for particular cases n=4 & n=3. Here very important the order because in the proof of Cauchy inequality in this cases we use simple but important ideas.

$$n=4.\ \frac{a_1+a_2+a_3+a_4}{4} = \frac{\frac{a_1+a_2}{2}+\frac{a_2+a_3}{2}}{2} \ge \frac{\sqrt{a_1a_2}+\sqrt{a_3a_4}}{2} \ge \sqrt{\sqrt{a_1a_2}\cdot\sqrt{a_3a_4}} \ge \sqrt[4]{a_1a_2a_3a_4}$$

$$n=3.\ \frac{a_1+a_2+a_3}{3} = \frac{4}{3}\cdot(a_1+a_2+a_3)\cdot\frac{1}{4} = \frac{a_1+a_2+a_3+\frac{a_1+a_2+a_3}{3}}{4} \ge \sqrt[4]{a_1a_2a_3\cdot\frac{a_1+a_2+a_3}{3}}$$

by Cauchy uneqaulity for n=4. By straightforward division by

$$\sqrt[4]{\frac{a_1+a_2+a_3}{3}}$$ we get $$\left(\frac{a_1+a_2+a_3}{3}\right)^{3/4} \ge (a_1a_2a_3)^{1/4} \Leftrightarrow \frac{a_1+a_2+a_3}{3} \ge \sqrt[3]{a_1a_2a_3}$$

It's possible another proof of Cauchy inequality when n=3 by using inequality $x^3 + y^3 \ge x^2y + xy^2$. Actually,

$$x^3 + y^3 + z^3 = \frac{x^2y+xy^2}{2}+\frac{y^2z+yz^2}{2}+\frac{z^2x+zx^2}{2} \ge xyz + xyz + xyz = 3xyz.$$

By setting $x=\sqrt[3]{a_1}$, $y=\sqrt[3]{a_2}$, $z=\sqrt[3]{a_3}$ we obtain $\frac{a_1+a_2+a_3}{3} \ge \sqrt[3]{a_1a_2a_3}$

Notice that from inequality $G_1 \ge G_0$ for any n it immediately follows $G_0 \ge G_{-1}$ for positive numbers (to do it is sufficient to use Cauchy inequality for numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$). We shall stay by inequality $G_2 \ge G_1$.

For n=4 we've $$\sqrt{\frac{a_1^2+a_2^2+a_3^2+a_4^2}{4}} = \sqrt{\frac{\left(\sqrt{\frac{a_1^2+a_2^2}{2}}\right)^2+\left(\sqrt{\frac{a_3^2+a_4^2}{2}}\right)^2}{2}} \ge \frac{\sqrt{\frac{a_1^2+a_2^2}{2}}+\sqrt{\frac{a_3^2+a_4^2}{2}}}{2}$$

$$\ge \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2} = \frac{a_1+a_2+a_3+a_4}{4}.$$ So, we obtain the inequality:

$$\sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{4}} \geq \frac{a_1 + a_2 + a_3 + a_4}{4}$$

For n=3 inequality $G_2 \geq G_1$ we could prove directly by squaring the both of inequality sides:

$$\sqrt{\frac{a_1^2+a_2^2+a_3^2}{3}} \geq \frac{a_1+a_2+a_3}{3} \Leftrightarrow 3\cdot(a_1^2+a_2^2+a_3^2) \geq (a_1+a_2+a_3)^2 \Leftrightarrow a_1^2+a_2^2+a_3^2 \geq$$

$$\geq a_1a_2 + a_2a_3 + a_3a_1 \Leftrightarrow \frac{a_1^2+a_2^2}{2} + \frac{a_2^2+a_3^2}{2} + \frac{a_3^2+a_1^2}{2} \geq a_1a_2 + a_2a_3 + a_3a_1.$$

But we shall show that this inequality we can prove by the way we've proved above for n=4 - by repeating way we've used in proof of Cauchy inequality for n=3 from inequality for n=4. This way in what follows we call by reveres step.

Let $G_2 = \sqrt{\frac{a_1^2+a_2^2+a_3^2}{3}}$. Then, $G_2 = \sqrt{\frac{3\cdot G_2^2+G_2^2}{4}} = \sqrt{\frac{a_1^2+a_2^2+a_3^2+G_2^2}{4}} \geq \frac{a_1+a_2+a_3+G_2}{4}$

$$\Leftrightarrow 4\cdot G_2 \geq a_1 + a_2 + a_3 + G_2 \Leftrightarrow 3G_2 \geq a_1+a_2+a_3 \Leftrightarrow G_2 \geq G_1.$$

Follows two exercises allows us convince once again in effective ideas of doubling and reverse step.

Exercise 9. Prove inequalities:

a) $\frac{\sin\alpha + \sin\beta}{2} \leq \sin\frac{\alpha+\beta}{2}$

b) $\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 + \sin\alpha_4}{4} \leq \sin\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}$

c) $\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{3} \leq \sin\frac{\alpha_1+\alpha_2+\alpha_3}{3}$

Exercise 10. Prove the truth of unequalities:

a) $\sin\alpha\cdot\sin\beta \leq \sin^2\frac{\alpha+\beta}{2}$

b) $\sin\alpha_1\cdot\sin\alpha_2\cdot\sin\alpha_3\cdot\sin\alpha_4 \leq \sin^4\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}$

c) $\sin\alpha_1\cdot\sin\alpha_2\cdot\sin\alpha_3 \leq \sin^3\frac{\alpha_1+\alpha_2+\alpha_3}{3}$

Exercise 11. Prove for any positive numbers a, b, c inequality:

$(a+b+c)\cdot(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9$ (it's desirable to do it by inequality $\frac{a+b}{2} \geq \sqrt{ab}$, no using $G_1 \geq G_0$ for three numbers).

Exercise 12. Prove for any positive a, b, c

a) $\sqrt{\frac{a+b}{c}} + \sqrt{\frac{b+c}{a}} + \sqrt{\frac{a+c}{b}} \geq 3\sqrt{2}.$

b) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}$

Exercise 13. Prove for any non-negative a,b,c:

$$a^3 + b^3 + c^3 \le 2\cdot(a+b+c)\cdot(a^2+b^2+c^2)$$

Exercise 14. Prove that for any triangle does hold inequality $S \ge 3\sqrt{3}\cdot r^2$, where S is area of this triangle and r is radius of inscribed circle into triangle. When does equality hold ?

Exercise 15. Let h_a, h_b, h_c are height of triangle dropped to the sides a, b, c respectively and d_a, d_b, d_c are distance from arbitrary point inside triangle to its sides a, b, c respectively. Prove that

$$h_a h_b h_c \ge 27\cdot d_a d_b d_c$$

When does equality hold ?

Exercise 16. Find out what's greater:

a) $2\cdot\sqrt[4]{0.9} + \sqrt[4]{1.05} + \sqrt[4]{1.1}$ or 4.

b) $\sqrt[4]{1.2} + \sqrt[4]{1.2} + \sqrt[4]{0.8} + 2\cdot\sqrt[4]{0.95}$ or 5.

c) $\log_4 5 + \log_5 6 + \log_6 7 + \log_7 8$ or 4.4

(As long as in further we shall frequently use mathematic induction method then in follows exercises we offer to reader perform uncomplicated but of course useful work to get some very important inequalities. Those inequalities in further will turn out as a particular cases of some generalized inequalities. You can ask here: why don't we prove some general theorems and drop particular cases ? Wouldn't it be easy ? The answer to this question is "yes" if the main point of this article would be a general results as it occur in mathematic science articles. However the main point of this article is not only concrete results. But what is more important here is to show appearance and developments of ideas, their interaction and correlations and different performance technic. Moreover, it's desirable that notwithstanding with preparation, just aspiration to master on this very interesting region of mathematics knowledge would turn out as a decisive factor which is defined the resoluteness of work with this article).

Exercise 17. Prove for any natural number n and real x>-1 truth of follows inequalities:

a) $(1+x)^n \ge 1 + n\cdot x$ (induction by n).

b) $(1+x)^{1/n} \le 1 + \frac{x}{n}$.

When is equality possible ?

Exercise 18.

a) Let $s, t \ge 0$ & $s+t=1$. Prove that for any natural number n inequality $s^n + t^n \ge \frac{1}{2^{n-1}}$ (induction by n). When is equality possible ?

b) Prove that $\left(\frac{a^n+b^n}{2}\right)^n \geq \frac{a+b}{2}$ for $a,b\geq 0$.

(Hint. Assume $s = \frac{a}{a+b}$ & $t = \frac{b}{a+b}$ for $a,b>0$ from exercise 12).

Exercise 19. Prove inequalities:

a) $\left(\frac{a_1^n+a_2^n+a_3^n+a_4^n}{4}\right)^{1/n} \geq \frac{a_1+a_2+a_3+a_4}{4}$

b) $\left(\frac{a_1^n+a_2^n+a_3^n}{3}\right)^{1/n} \geq \frac{a_1+a_2+a_3}{3}$ (Use reverse step).

We are going to prove Cauchy's inequality (general case) and we shall do that by several different ways and each one has special interest.

Theorem 1. For any non-negative numbers $a_1, a_2, \ldots, a_n$ holds inequality

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n} \qquad (8)$$

which becomes an equality only if $a_1=a_2=\ldots=a_n$.

Proof: (induction by n).

1. *Base.* When n=2 the theorem is true as we've proved.
2. *Induction.*

Suppose that for any $2\leq k<n$ and any set of k non-negative numbers

$b_1, b_2, \ldots, b_k$ holds inequality $\frac{b_1+b_2+\ldots+b_k}{k} \geq \sqrt[k]{b_1 b_2 \ldots b_k}$ and an

equality holds only when $b_1=b_2=\ldots=b_k$. Prove that for arbitrary set of k non-negative numbers $a_1, a_2, \ldots, a_n$ holds inequality (8).

Consider two cases:

1. $n=2\cdot m$, then $m<n$ and by supposition of induction when $k=m$ and by base we get

$$\frac{a_1+a_2+\ldots+a_{2m}}{2m} = \frac{\frac{a_1+a_2+\ldots+a_m}{m} + \frac{a_{m+1}+\ldots+a_{2m}}{m}}{2} \geq \frac{\sqrt[m]{a_1 a_2 \ldots a_m} + \sqrt[m]{a_{m+1}\ldots a_{2m}}}{2} \geq$$

$$\geq \sqrt{\sqrt[m]{a_1 a_2 \ldots a_m}\cdot\sqrt[m]{a_{m+1}\ldots a_{2m}}} = \sqrt[2m]{a_1 a_2 \ldots a_{2m}}$$

An equality is possible if and only if $a_1=a_2=\ldots=a_m$ и $a_{m+1}=\ldots=a_{2m}$ &

$\sqrt[m]{a_1 a_2 \ldots a_m} = \sqrt[m]{a_{m+1}\ldots a_{2m}}$, and it equivalents $a_1=a_2=\ldots=a_{2m}$.

2. $n=2\cdot m - 1$. But $n+1=2m$ & $m<n$. Then by supposition considered before 1-st case holds inequality (8) for the set of 2m numbers $a_1, a_2, \ldots, a_n, \frac{a_1+a_2+\ldots+a_n}{n}$

Denote $S = \dfrac{a_1+a_2+\ldots+a_n}{n}$, we get

$$S = \frac{n\cdot S_1+S}{n+1} = \frac{a_1+a_2+\ldots+a_n+S}{n+1} \geq \sqrt[n+1]{a_1a_2\ldots a_n\cdot S} \iff S^{n+1} \geq a_1a_2\ldots a_n\cdot S \iff$$

$\iff S\cdot(S^n-a_1a_2\ldots a_n) \geq 0$. If at least one of numbers $a_1,\ldots,a_n$ more then 0, then $S>0$ and we get $S^n \geq a_1a_2\ldots a_n \iff S \geq \sqrt[n]{a_1a_2\ldots a_n}$, and an equality holds if and only if $a_1=a_2=\ldots=a_n=S \iff a_1=a_2=\ldots=a_n$.
Theorem is proved. ■

Exersice 20. Prove inequalities for positive numbers $a_1,a_2,\ldots,a_n$:

a) $\dfrac{a_1}{a_2} + \dfrac{a_2}{a_3} + \ldots + \dfrac{a_n}{a_1} \geq n$

b) $\sqrt[n]{a_1a_2\ldots a_n} \geq \dfrac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}$ $\quad (G_0 \geq G_{-1})$

c) $(a_1+a_2+\ldots a_n)\cdot\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}\right) \geq n^2$.

*Exercise 21**. For non-negative numbers $x_1,x_2,\ldots,x_n$ prove inequality

$$(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_n) \geq (1+\sqrt[n]{x_1x_2\ldots x_n})^n$$

Exercise 22. For any natural number n prove follows inequalities

a) $n! \leq \left(\dfrac{n}{2}+1\right)^{n-1}$

b) $\sqrt[n]{n} - 1 \leq \dfrac{2}{\sqrt{n}}$

c) $n\cdot\sqrt[n]{n+1} - 1 \leq 1+\dfrac{1}{2}+\ldots+\dfrac{1}{n} \leq 1 + n\cdot\left(1-\dfrac{1}{\sqrt[n]{n}}\right)$

d) $\left(1+\dfrac{1}{4}\right)\cdot\left(1+\dfrac{1}{8}\right)\cdot\ldots\cdot\left(1+\dfrac{1}{2^n}\right) < 2$

Exercise 23. Let $f(n) = \left(1+\dfrac{1}{n}\right)^n$ & $g(n) = \left(1+\dfrac{1}{n}\right)^{n+1}$. Prove that for any $n\geq1$: $f(n+1) > f(n)$ & $g(n+1) > g(n)$.

Theorem 2. (Ellers). Let n positive numbers $x_1,x_2,\ldots,x_n$ satisfy the condition $x_1\cdot x_2\cdot\ldots\cdot x_n = 1$, then $x_1 + x_2 + \ldots + x_n \geq n$, and an equality holds if and only if $x_1=x_2=\ldots=x_n$.

Proof. (Induction by n):

1. *Base.* $n = 2$; $x_1,x_2>0$ & $x_1x_2=1$, then $x_1+x_2\geq2 \iff x_1+x_2\geq2\sqrt{x_1x_2} \iff (\sqrt{x_1}-\sqrt{x_2})^2 \geq 0$.

2. *Induction*. Suppose that the theorem is true for any $n\geq 2$. We're going to prove that the theorem is true for $n+1$. So, let $x_1, x_2, \ldots, x_{n+1} > 0$ & $x_1 x_2 \ldots x_n x_{n+1} = 1$. Here is impossible that all $x_i > 1$. Therefore, we can find two numbers such that one of them greater than unity and another smaller than unity. In what follows we shall suppose that $x_n \leq 1$ & $x_{n+1} \geq 1$. (Always we can denote numbers $x_1, x_2, \ldots, x_{n+1}$ so that exactly two of last numbers would satisfy such condition. Then for set of n numbers $x_1, x_2, \ldots, x_{n-1}, x_n x_{n+1}$ the theorem is true, i.e. $x_1 \cdot x_2 \ldots \cdot (x_n x_{n+1})$ it follows that

$$x_1 + x_2 + \ldots + x_{n-1} + x_n \cdot x_{n+1} \geq n \iff x_1+x_2+\ldots+x_{n-1}+x_n+x_{n+1} \geq$$

$$\geq n+1 - (1-x_n-x_{n+1}+x_n \cdot x_{n+1}) \iff x_1+x_2+\ldots+x_n+x_{n+1} \geq n+1 - (1-x_n)\cdot(1-x_{n+1}).$$

But $(1-x_n)\cdot(1-x_{n+1}) \leq 0$. Therefore, $n+1 - (1-x_n)\cdot(1-x_{n+1}) \geq n+1$.

Thus, finally we get $x_1+x_2+\ldots+x_n+x_{n+1} \geq n+1$.

Let $x_1+x_2+\ldots+x_{n+1} = n+1 \Rightarrow n+1 - (1-x_n)\cdot(1-x_{n+1}) = n + 1 \iff \left[\begin{array}{l} x_n = 1 \\ x_{n+1} = 1 \end{array}\right.$.

Suppose that $x_{n+1} = 1$, then $x_1+x_2+\ldots+x_n = n$ and it means that by supposition of induction $x_1 = \ldots = x_n = 1 = x_{n+1}$. Theorem is proved. ■

Cauchy's inequality (8) we could get as the simple *corollary* from Theorem 2. Let $a_1, a_2, \ldots, a_n > 0$ then by notation $\sqrt[n]{a_1 a_2 \ldots a_n}$ by p we get for numbers $\frac{a_1}{p}, \frac{a_2}{p}, \ldots, \frac{a_n}{p}$ inequality $\frac{a_1}{p} + \frac{a_2}{p} + \ldots + \frac{a_n}{p} \geq n$, i.e. $\frac{a_1}{p} \cdot \frac{a_2}{p} \cdot \ldots \cdot \frac{a_n}{p} =$ $= \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{p^n} = 1$. Hence, $\frac{a_1+a_2+\ldots+a_n}{n} \geq p$. If $\frac{a_1}{p} = \frac{a_2}{p} = \ldots = \frac{a}{p} = 1$, i.e. when $a_1 = a_2 = \ldots = a_p$ the inequality turns to equality and only in this case.

*Exercise 24**. Let $f(x) = a\cdot x^2 + b\cdot x + c$, $a,b,c>0$ & $a+b+c=1$. Prove that for any n positive numbers $x_1, x_2, \ldots, x_n$ such that $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 1$ holds inequality:

$$f(x_1)\cdot f(x_2)\cdot \ldots \cdot f(x_n) \geq 1$$

And equality is possible if only if $x_1 = x_2 = \ldots = x_n$ (*Hint. Prove indentity*

$$f(x_1)\cdot f(x_2) = f(x_1 x_2) - ab\cdot x_1 x_2 \cdot (1-x_1)\cdot(1-x_2) - bc\cdot(1-x_1)\cdot(1-x_2) - ac\cdot(1-x_1^2)\cdot(1-x_2^2)$$

for any positive x_1 *and* x_2).

Exercise 25. For any set of positive numbers $a_1, a_2, \ldots, a_n$ prove inequality:

$$\frac{a_1-a_3}{a_2+a_3} + \frac{a_2-a_4}{a_3+a_4} + \ldots + \frac{a_{n-2}-a_n}{a_{n+1}+a_n} + \frac{a_{n-1}-a_1}{a_n+a_1} + \frac{a_n-a_2}{a_1+a_2} \geq 0$$

Consider next problems. Find the greatest value of product $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ if $x_1+x_2+\ldots x_n=a > 0$ & $x_1, x_2, \ldots, x_n \geq 0$.

By Cauchy's inequality $\frac{a}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$ and the upper bound $\frac{a}{n}$ for $\sqrt[n]{x_1 x_2 \ldots x_n}$ could be reached when $x_1=x_2=\ldots=x_n=\frac{a}{n}$. Thus, the greatest value of $x_1 x_2 \ldots x_n$ is equal to $(\frac{a}{n})^n$.

What would we do if we couldn't know the Cauchy's inequality ? But we still have to solve the problem and suppose that we already have solved it. Then, obviously that Cauchy's inequality would be a *corollary* from this problem and we'd have one more way to prove Cauchy's inequality. So matter is to solve set above problems without using Cauchy's inequality.

Denote by $f_n(a)$ the greatest value of product $x_1 x_2 \ldots x_n$ over all ordered sets $(x_1, x_2, \ldots, x_n)$ of real numbers such that $x_1, x_2, \ldots, x_n \geq 0$ & $x_1+x_2+\ldots+x_n = a$. The set of such sets we shall note by $D_n(a)$.
The correspondence $(x_1, x_2, \ldots, x_n) \mapsto (tx_1, tx_2, \ldots, tx_n)$ between sets $D_n(a)$ & $D_n(ta)$ is single-valued. Therefore, $f_n(ta) = t^n \cdot f_n(a)$.

Really, any set from $D_n(ta)$ we can rewrite like this $(tx_1, tx_2, \ldots, tx_n)$, where $(x_1, x_2, \ldots, x_n) \in D_n(a)$ and it follows

$$f_n(ta) = \max\ (tx_1)\cdot(tx_2)\cdot\ldots\cdot(tx_n) = t^n \cdot \max\ x_1 x_2 \ldots x_n,$$

where $(x_1, x_2, \ldots, x_n)$ gets all values from $D_n(a)$.

Hence, $f_n(ta) = t^n \cdot f_n(a)$, and in particular, $f_n(a) = a^n \cdot f_n(1)$.

Transfor the initial system of restrictions defined $D_n(a)$

$$\begin{cases} x_1+x_2+\ldots+x_n=a \\ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \end{cases} \Leftrightarrow \begin{cases} x_1+x_2+\ldots+x_{n-1}=a-x_n \\ x_1, x_2, \ldots, x_{n-1} \geq 0 \\ 0 \leq x_n \leq a \end{cases} \Leftrightarrow \begin{cases} \frac{x_1}{a}+\frac{x_2}{a}+\ldots+\frac{x_n}{a}=1-\frac{x_n}{a} \\ x_1, x_2, \ldots, x_{n-1} \geq 0 \\ 0 \leq \frac{x_n}{a} \leq 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 \leq s < 1 \\ s_1+s_2+\ldots+s_{n-1}=s \\ s_1, s_2, \ldots, s_{n-1} \geq 0 \\ x_i = as_i,\ i=1,2,\ldots,n-1 \\ x_n = a(1-s) \end{cases}$$. Then,

$$f_n(a) = \max_{D_n(a)} x_1 x_2 \ldots x_n = \max_{s\in[0,1]} \{\max_{D_{n-1}(s)} (as_1)\cdot(as_2)\cdot\ldots\cdot(as_{n-1})\cdot a(1-s)\} =$$

$$= \max_{s\in[0,1]} \{a^n (1-s)\cdot \max\ s_1 s_2 \ldots s_{n-1}\} = a^n \cdot \max_{s\in[0,1]} (1-s)\cdot f_{n-1}(s) =$$

$$= a^n \cdot \max_{s\in[0,1]} \{(1-s)\cdot s^{n-1}\cdot f_{n-1}(1)\} = a^n \cdot f_{n-1}(1) \cdot \max_{s\in[0,1]} (1-s)\cdot s^{n-1}.$$

So, now our problem is to find a greatest value of function

$h(s) = (1-s)\cdot s^{n-1}$ onto segment $[0,1]$. $h'(s)=(n-1)\cdot s^{n-2}-n\cdot s^{n-1}=0 \Leftrightarrow \left[\begin{array}{l} s=0 \\ s=\frac{n-1}{n}\end{array}\right.$

Hence, $\max\limits_{s\in[0,1]} h(s) = \max\{h(0),h(1),h(\frac{n-1}{n})\}=(\frac{n-1}{n})^{n-1}\cdot\frac{1}{n}=\frac{(n-1)^{n-1}}{n^n}=h(\frac{n-1}{n})$.

So, $f_n(1) = \frac{(n-1)^{n-1}}{n^n}\cdot f_{n-1}(1)$. Obviously, $f_1(1) = 1$. Since $n^n\cdot f_n(1) =$
$= (n-1)^{n-1}\cdot f_{n-1}(1)$, then $n^n\cdot f_n(1) = 1^1\cdot f_1(1) \Rightarrow f_n(1)=\frac{1}{n^n}$. Hence, $f_n(a)=(\frac{a}{n})^n$

Thus, $x_1x_2\ldots x_n\le(\frac{a}{n})^n$ and equality could be reached if & only if $x_n=\frac{a}{n}$.

But then $x_1x_2\ldots x_{n-1}\le(\frac{a}{n})^{n-1}$ and since $x_1x_2\ldots x_{n-1}=a\cdot\frac{n-1}{n}=a_1$, then,

$x_1x_2\ldots x_{n-1}\le(\frac{a_1}{n-1})^{n-1}$ and $x_{n-1}=\frac{a_1}{n-1}=\frac{a\cdot(n-1)}{n\cdot(n-1)}=\frac{a}{n}$ and so on.

(This moment you could prove by induction by n).

Cauchy's inequality by itself and also some inequalities-corollaries from its are very strong tool over many regions in mathematics on solution of problems and proof of theorems.

Assume in Cauchy's inequality $a_1=a_2=\ldots=a_m=1+x$, where $a_{m+1}=\ldots=a_n=1$, then we get:

$$\sqrt[n]{(1+x)^m}\le\frac{m\cdot(1+x)+n-m}{n}=1+\frac{m}{n}\cdot x \quad\text{or}\quad (1+x)^{m/n}\le 1+\frac{m}{n}\cdot x.$$

(equality is possible only if $x=0$). So, for any rational r which is smaller than unity holds inequality

$$(1+x)^r\le 1+r\cdot x \qquad (10) \quad - \textit{Bernoulli's inequality}$$

If we denote $1+x$ by t, then Bernoulli's inequality can be written

$$t^r\le 1+r\cdot(t-1) \Leftrightarrow t^r-r\cdot t+r-1\le 0.$$

Let $r\in\mathbb{Q}$ (r-rational number) & $r>1$, then $\frac{1}{r}<1$ and for any $s>0$ holds inequality $s^{1/r}-\frac{1}{r}\cdot s+\frac{1}{r}-1\le 0$. If for any $t>0$ instead of s we substitute t^r, then we get inequality

$$t-\frac{1}{r}\cdot t^2+\frac{1}{r}-1\le 0 \Leftrightarrow t^r-r\cdot t+r-1\le 0.$$

For any $x>-1$ instead of t we substitute $x+1>0$ and we obtain inequality

$$(1+x)^r\ge 1+r\cdot x,\ (r>1)$$

Thus, we've got two pairs of very important in applications inequalities:

$$(1+x)^r\ge 1+r\cdot x,\quad x>-1,\ r\in\mathbb{Q}\ \&\ r>1 \qquad (11)$$

$$(1+x)^r\le 1+r\cdot x,\quad x>-1,\ r\in\mathbb{Q}\ \&\ 0<r<1 \qquad (12)$$

(Equality in two of inequalities is possible only if $x=0$)

$$x^r - r\cdot x + r - 1 \geq 0, \quad x>0,\ r\in\mathbb{Q}\ \&\ r>1 \qquad (13)$$

$$x^r - r\cdot x + r - 1 \leq 0, \quad x>0,\ r\in\mathbb{Q}\ \&\ 0<r<1 \qquad (14)$$

(Equality in two of inequalities is possible only if $x=1$).

Using of rational exponent is restriction which we can simply remove by some means of mathematics analysis. If we stay on conceptions of limits and continuity of exponential function, then extension inequalities (11),(12),(13),(14) on real numbers we can carry out by the same way. Sufficient to do it for anyone of them. Inequality (14) by historical causes convenient to this role as base. We shall get others as a corollary from its.

Let α - real number with $0<\alpha<1$. Then there exist a sequence of rational numbers $r_1, r_2, \ldots, r_n, \ldots$, such that $\alpha = \lim_{n\to\infty} r_n$ (sequence of rational approximations of the real number α) & $0<r_n<1$. But then for any $n\in\mathbb{N}$ holds inequality $x^{r_n} - r_n\cdot x + r_n - 1 \leq 0$. By passage to the limits when $n\to\infty$:

$$\lim_{n\to\infty} (x^{r_n} - r_n\cdot x + r_n - 1) \leq 0 \Leftrightarrow x^{\lim_{n\to\infty} r_n} - x\cdot\lim_{n\to\infty} r_n + \lim_{n\to\infty} r_n - 1 \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x^\alpha - \alpha\cdot x + \alpha - 1 \leq 0.$$

But by passaging to the limits the strict inequality $x>0$ & $x\neq 1$ turns to weak inequality. We have to correct it. It's easy.
Let r - rational numbers, then $\alpha<r<1$ (there exist such number - prove that *between any two real numbers there exist rational number*).
Then $0<\frac{\alpha}{r}<1$. Denote $\frac{\alpha}{r}$ by β. We get:

$x^\alpha - \alpha\cdot x + \alpha - 1 = (x^\beta)^r - r\cdot\beta x + r\cdot\beta - 1$. Let $x\neq 1$, then $x^\beta\neq 1$ and for x^β and rational $0<r<1$ holds inequality $(x^\beta)^r - r\cdot\beta x + r - 1 < 0$.

Hence, $(x^\beta)^r - r\cdot\beta x + r\cdot\beta - 1 < r\cdot x^\beta - r\cdot\beta x + r\cdot\beta - r$.

But for any real exponent $0<\beta<1$ holds inequality

$x^\beta - \beta\cdot x + \beta - 1 \leq 0$. Therefore, finally we obtain:

$x^\alpha - \alpha\cdot x + \alpha - 1 < r\cdot(x^\beta - \beta\cdot x + \beta - 1) \leq 0$. What's needed to prove.

By involving powerful means of analysis - derivatives we'd sharply reduce technical work, but this economy is under sign of question since a lot of theory representing this possibilities suggest to spend enormous efforts that make sense only if the work on mastering this theory already has done well. And the goal of its using does not restrict one problem. Suppose that last condition is true, then we're going to prove inequality

$$x^\alpha - \alpha\cdot x + \alpha - 1 \leq 0 \quad \text{for} \quad x>0\ \&\ 0<\alpha<1.$$

Consider function $f(x) = x^\alpha - \alpha\cdot x + \alpha - 1 \Rightarrow f'(x) = \alpha\cdot x^{\alpha-1} - \alpha$.

Equation $f'(x)=0 \Leftrightarrow x^{\alpha-1}-1=0$ has a unique solution: $x=1$.

For $0<x<1$ $x^{\alpha-1}-1>0$, and for $x>1$ $x^{\alpha-1}-1<0$ since $0<\alpha<1$. Therefore when $x=1$ $f(x)$ reach the maximum $f(1)=0$ and it means that for any $x>0$

$$x^\alpha - \alpha\cdot x + \alpha - 1 \leq 0,$$

and the equality occur only if $x=1$.

So we have for now four following inequalities:

$$(1+x)^{\alpha} \geq 1 + \alpha\cdot x, \qquad x>-1 \ \&\ \alpha>1 \qquad (15)$$

equality occur if x=0

$$(1+x)^{\alpha} \leq 1 + \alpha\cdot x, \qquad x>-1 \ \&\ 0<\alpha<1 \qquad (16)$$

$$x^{\alpha} - \alpha\cdot x + \alpha - 1 \geq 0, \qquad x>0 \ \&\ \alpha>1 \qquad (17)$$

equality occur if x=1

$$x^{\alpha} - \alpha\cdot x + \alpha - 1 \leq 0, \qquad x>0 \ \&\ 0<\alpha<1 \qquad (18)$$

Exercise 26. a) Prove *Holder's inequalities*:

$$s^{1/p}\cdot t^{1/q} \leq \frac{s}{p}+\frac{t}{q}, \quad p>1 \quad \& \quad \frac{1}{p}+\frac{1}{q}=1$$

$$s^{1/p}\cdot t^{1/q} \geq \frac{s}{p}+\frac{t}{q}, \quad 0<p<1 \quad \& \quad \frac{1}{p}+\frac{1}{q}=1 \qquad s,t>1$$

Hint. In inequalities 17 & 18 make substitutions $x=\frac{s}{t}$, $p=\frac{1}{\alpha}$, $q=\frac{1}{1-\alpha}$.

b) prove inequalities and find out when occur equality:

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^{p}\right)^{1/p}\cdot\left(\sum_{i=1}^{n} y_i^{q}\right)^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1, \ p>1$$

$$\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^{p}\right)^{1/p}\cdot\left(\sum_{i=1}^{n} y_i^{q}\right)^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1, \ 0<p<1$$

variables $x_1,x_2,\ldots,x_n$ & $y_1,y_2,\ldots,y_n$ are non-negative; in second inequality $y_1,y_2,\ldots,y_n>0$ since $q=\frac{p}{p-1}<0$.

(*Hint. In Holder's inequalities assume:* $s_i=\frac{x_i^{p}}{\sum_{i=1} x_i^{p}}$, $t=\frac{y_i^{q}}{\sum_{i=1} y_i^{q}}$, $i=1,\ldots,n$)

Those inequalities as well are calling by generalized Gelder's inequalities.

c) Prove *Minkovsky's inequalities*. Find out when equality occur.

$$\left(\sum_{i=1}^{n}(x_i+y_i)^{p}\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^{p}\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^{p}\right)^{1/p}, \text{ if } p>1$$

$$\left(\sum_{i=1}^{n}(x_i+y_i)^{p}\right)^{1/p} \geq \left(\sum_{i=1}^{n} x_i^{p}\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^{p}\right)^{1/p}, \quad \text{if } 0<p<1.$$

Hint. Use indentity:

$$\sum_{i=1}^{n}(x_i+y_i)^{p} = \sum_{i=1}^{n} x_i\cdot(x_i+y_i)^{p-1} + \sum_{i=1}^{n} y_i\cdot(x_i+y_i)^{p-1}$$

And use to each one of sums in right side the generalized Gelder's inequality with exponents p & q, $q=\frac{p}{p-1}$.

The inequalities from exercise 26 are very important in mathematics. As regard Minkovsky's inequality is an object for another talking about generalization of distance concept.

Let t_1 & t_2 - positive real numbers, then substitution $x=\frac{t_1}{t_2}$ into inequality (18) yields:

$$\left(\frac{t_1}{t_2}\right)^{\alpha} - \alpha\cdot\frac{t_1}{t_2} + \alpha - 1 \le 0 \quad\Leftrightarrow\quad t_1^{\alpha}\cdot t_2^{\alpha-1} - \alpha\cdot t_1 - (1-\alpha)\cdot t_2 \le 0.$$

And equality occur only if $t_1/t_2=1 \Leftrightarrow t_1=t_2$

Assume $\alpha_1=\alpha$ & $\alpha_2=1-\alpha$, and in last inequality we get that for any $t_1,t_2>0$ & $\alpha_1,\alpha_2\ge0$ & $\alpha_1+\alpha_2=1$ holds inequality:

$$t_1^{\alpha_1}\cdot t_2^{\alpha_2} \le \alpha_1\cdot t_1 + \alpha_2\cdot t_2 \qquad (19)$$

Equality occur only if $t_1=t_2$.

Theorem 3. For any numbers $x_1,x_2,\ldots,x_n\ge0$ & $\alpha_1,\alpha_2,\ldots,\alpha_n\ge0$ such that $\alpha_1+\alpha_2+\ldots+\alpha_n=1$ holds inequality:

$$x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n} \le \alpha_1\cdot x_1 + \alpha_2\cdot x_2 + \ldots + \alpha_n\cdot x_n \qquad (20)$$

(Equality occur only if $x_1=x_2=\ldots=x_n$).

Numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$ are calling *weights*.

Numbers $\alpha_1\cdot x_2 + \alpha_2\cdot x_2 + \ldots + \alpha_n\cdot x_n$ & $x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n}$ called *weighted arithmetic and geometric average* respectively.

Proof: (induction by n).

1. Base. We already have this (this is inequality (19)).
2. Induction. Suppose that the theorem is true for any $n\ge2$ and we have to prove that the theorem is true for $n+1$.

Let $x_1,x_2,\ldots,x_{n+1}\ge0$ & $\alpha_1,\alpha_2,\ldots,\alpha_{n+1}\ge0$ & $\alpha_1+\alpha_2+\ldots\alpha_n=1$.

Suppose that $\alpha_{n+1}\ne0$, since otherwise we obviously have the induction.

Therefore,

$$\frac{\alpha_1}{1-\alpha_{n+1}} + \frac{\alpha_2}{1-\alpha_{n+1}} + \ldots + \frac{\alpha_n}{1-\alpha_{n+1}} = \frac{\alpha_1+\alpha_2+\ldots+\alpha_n}{1-\alpha_{n+1}} = \frac{1-\alpha_{n+1}}{1-\alpha_{n+1}} = 1.$$

Use consequently inequality (19) (The Base of induction) to numbers

$x_{n+1}, (x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n})^{\frac{1}{1-\alpha_{n+1}}}$ with weights α_{n+1} & $1-\alpha_{n+1}$ and supposition of induction to numbers $x_1,x_2,\ldots,x_n$ with weights $\frac{\alpha_1}{1-\alpha_{n+1}},\ldots,\frac{\alpha_n}{1-\alpha_{n+1}}$ we get:

$$x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n}\cdot x_{n+1}^{\alpha_{n+1}} = x_{n+1}^{\alpha_{n+1}}\cdot\left[(x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n})^{\frac{1}{1-\alpha_{n+1}}}\right]^{1-\alpha_{n+1}} \le \alpha_{n+1}\cdot x_{n+1} +$$

$$+ (1-\alpha_{n+1})\cdot x^{\frac{\alpha_1}{1-\alpha_{n+1}}}\cdot x^{\frac{\alpha_2}{1-\alpha_{n+1}}}\cdot\ldots\cdot x^{\frac{\alpha_n}{1-\alpha_{n+1}}} \le (1-\alpha_{n+1})\cdot(\frac{\alpha_1}{1-\alpha_{n+1}}\cdot x_1+\frac{\alpha_2}{1-\alpha_{n+1}}\cdot x_2+\ldots$$

$$+\frac{\alpha_n}{1-\alpha_{n+1}}\cdot x_n) + \alpha_{n+1}\cdot x_{n+1} = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n .$$

Equality occur if $x_{n+1}=(x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n})^{1/1-\alpha_{n+1}}$ & $x_1=x_2=\ldots=x_n$,

i.e. when $x_1=x_2=\ldots=x_{n+1}$. ■

We could prove this theorem by another way, namely as a directly corollary from Cauchy's inequality. First of all prove it for rational weights: $r_1,r_2,\ldots,r_n$. So, let $r_1,r_2,\ldots,r_n$ are non-negative rational numbers such that $r_1+r_2+\ldots+r_n=1$, then they could be written as fractions with the same denominator: $r_i=\frac{k_i}{m}$, $i=1,2,\ldots,n$ and in this case $k_1+\ldots+k_n=m$

$$x_1^{r_1}\cdot x_2^{r_2}\cdot\ldots\cdot x_n^{r_n} = \sqrt[m]{x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}} \le \frac{k_1x_1+k_2x_2+\ldots+k_nx_n}{m}$$

by Cauchy's inequality used to set of m numbers $t_1,t_2,\ldots,t_m$ such that

$t_1=t_2=\ldots=t_{k_1}=x_1$, $t_{k_1+1}=\ldots=t_{k_1+k_2}=x_2$, $\ldots$, $t_{k_1+\ldots+k_{n-1}+1}=\ldots=t_{k_1+\ldots+k_n}=x_n$

Equality in the derived inequality reach only if $x_1=x_2=\ldots=x_n$.
From proved inequality following fact. Let $x_1,x_2,\ldots,x_n\ge0$ are real numbers & $r_1,r_2,\ldots,r_n>0$ are rational then holds inequality

$$(x_1^{r_1}\cdot x_2^{r_2}\cdot\ldots\cdot x_n^{r_n})^{\frac{1}{r_1+r_2+\ldots+r_n}} \le \frac{r_1x_1+r_2x_2+\ldots+r_nx_n}{r_1+r_2+\ldots+r_n}$$

Let $p_1,p_2,\ldots,p_n>0$ real numbers then each of p_i $(i=1,\ldots,n)$ is a limit of some sequence of positive rational numbers, i.e.

$$p_i=\lim_{k\to\infty} r_{ik}\ ,\ i=1,2,\ldots,n,\ \text{with}\ r_{ik}>0\ \text{for all}\ k\in\mathbb{N}.$$

Pass in inequality

$$(x_1^{r_{1k}}\cdot x_2^{r_{2k}}\cdot\ldots\cdot x_n^{r_{nk}})^{\frac{1}{r_{1k}+\ldots+r_{nk}}} \le \frac{r_{1k}x_1+r_{2k}x_2+\ldots+r_{nk}x_n}{r_{1k}+r_{2k}+\ldots+r_{nk}}$$

of a limit when $k\to\infty$ we obtain inequality

$$\left(x_1^{p_1}\cdot x_2^{p_2}\cdot\ldots\cdot x_n^{p_n}\right)^{\frac{1}{p_1+\ldots+p_n}} \le \frac{p_1x_1+p_2x_2+\ldots+p_nx_n}{p_1+p_2+\ldots+p_n} \qquad (21)$$

$p_1, p_2, \ldots, p_n$ as before are calling weights.

Exercise 27.
a) Let a, b, x, y are positive real numbers. Prove inequality:

$$\left(\frac{x}{a}\right)^{a}\cdot\left(\frac{y}{b}\right)^{b} \le \left(\frac{x+y}{a+b}\right)^{a+b}$$

b) Let $a_1, a_2, \ldots, a_n$ & $x_1, x_2, \ldots, x_n$ are positive real numbers.

Prove inequality:

$$\left(\frac{x_1}{a_1}\right)^{a_1}\cdot\left(\frac{x_2}{a_2}\right)^{a_2}\cdot\ldots\cdot\left(\frac{x_n}{a_n}\right)^{a_n} \le \left(\frac{x_1+x_2+\ldots+x_n}{a_1+a_2+\ldots+a_n}\right)^{a_1+\ldots+a_n}$$

Exercise 28. Discuss function $f(t)=\frac{(1+t)^{1+p}}{t^p}$, where $p>0$ on extremum.

Prove inequality

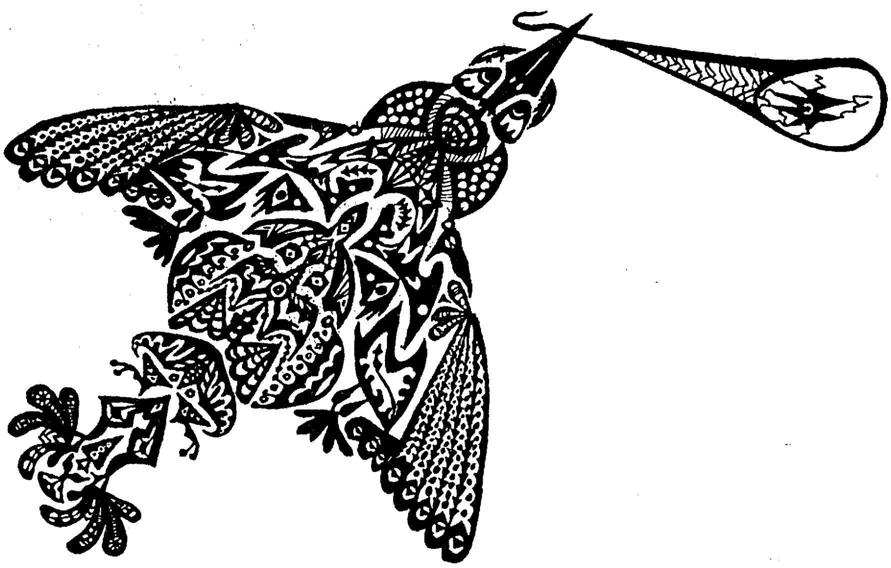
$$\frac{(1+p)^p}{p^p} \le \frac{(1+t)^{1+p}}{t^p}, \quad t>0 \qquad (22)$$

Equality occur only if $t=p$.

Prove by inequality (22) the inequalities from exercise 27. The inequality from exercise no. 23 is a base inequality since from this inequality follows but in reverse order all inequality considered above in this part of articles.

Exercise 29. Let numbers a, b, c - length of triangle sides.
Prove inequality:

$$\left(1+\frac{b-c}{a}\right)^{a}\cdot\left(1+\frac{c-a}{b}\right)^{b}\cdot\left(1+\frac{a-b}{c}\right)^{c} \le 1$$



Variation 3.

Next step is consist of introduction of interesting way which we are using more than twice and therefore we can call it by parametric method. However, actually we are talking about the method which is based on using of one or more parameters and which allows us to use some base inequalities in cases when directly using of them does not give desirable result. Had rather to explain the essence of the method is to show how to use it in some problems.
I start from the problem which led me to an idea of using of indefinite parameters:

Find the greatest value of function:

$$f(x,y,z) = \sqrt{4x+1} + \sqrt{4y+1} + \sqrt{4z+1}, \text{ если } x,y,z\geq 0 \text{ и } x+y+z=1$$

First of all I used Cauchy's inequality to expression $\sqrt{4a+1}$ represented as $\sqrt{(4a+1)\cdot 1}$. Then $\sqrt{4a+1} \leq \frac{4a+1+1}{2} = 2a+1$.

By this inequality we could get the upper bound for possible values of function $f(x,y,z)$:

$$f(x,y,z) \leq 2x+1 + 2y+1 + 2z+1 = 2\cdot(x+y+z) + 3 = 5.$$

But when I noticed that equality occur only if $x=y=z=0$ $(4a+1=1 \Leftrightarrow a=0)$, I understood that this way would give just unreachable upper bound for values of function $f(x,y,z)$, since the conditions $x+y+z=1$ and $x=y=z=0$ are uncompatible. Therefore I denied to use Cauchy's inequality and I went on another way, namely I've used inequality:

$\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \geq \frac{a+b+c}{3}$, where $a=\sqrt{4x+1}$, $b=\sqrt{4y+1}$, $c=\sqrt{4z+1}$. Then

$\left(\frac{f(x,y,z)}{3}\right)^2 \geq \frac{4x+1 + 4y+1 + 4z+1}{3} = 1+\frac{4}{3}\cdot(x+y+z) = \frac{7}{3} \Leftrightarrow f^2(x,y,z)\leq 21 \Leftrightarrow$

$\Leftrightarrow f(x,y,z)\leq\sqrt{21}$, and equality occur if $x=y=z=\frac{1}{3}$.

However, finishing with this problem I went back to the Cauchy's inequality $\frac{a+b}{2} \geq \sqrt{ab}$. Since this problem was offered on region mathematic olympiad then as I thought this problem should be solved as much as possible by scanty means. As I understand the Cauchy's inequality is completely satisfies demands moreover, that the function's form pushes me to this way. As long as I failed here because first partner $4a+1$ by Cauchy's inequality than, I decided to not fix the second factor under root, to be more precisely, instead of expression $\sqrt{4x+1} + \sqrt{4y+1} + \sqrt{4z+1}$ consider expression

$\sqrt{(4x+1)\cdot t} + \sqrt{(4y+1)\cdot t} + \sqrt{(4z+1)\cdot t} = \sqrt{t}\cdot f(x,y,z)$.

Since $\sqrt{(4x+1)\cdot t} \leq \frac{4x+1+t}{2}$, $\sqrt{(4y+1)\cdot t} \leq \frac{4y+1+t}{2}$, $\sqrt{(4z+1)\cdot t} \leq \frac{4z+1+t}{2}$,

then

$$\sqrt{t}\cdot f(x,y,z) \leq \frac{4\cdot(x+y+z)+3+3t}{2} = \frac{7+3t}{2} \Leftrightarrow f(x,y,z) \leq \frac{7+3t}{2\sqrt{t}}$$

and equality occur only if its holds in each of three inequalities it means that when $4x+1=4y+1=4z+1=t$ or when

$x=y=z=\frac{t-1}{4}$. Substitution x,y,z into condition $x+y+z=1$ yields the value of parameter $t=\frac{7}{3}$, and it gives the value of variables $x=y=z=\frac{1}{3}$.

Inequality $f(x,y,z)\le\frac{7+3t}{2\sqrt{t}}$ holds for any $x,y,z\ge0$ & $x+y+z=1$ and for any $t>0$ (and parameter t is free, i.e. it is not linked with variables x,y,z by any conditions). Therefore, in particular it holds for $t=7/3$

and it yields the equality $f(x,y,z)\le\frac{7+7}{2\sqrt{7/3}}=\sqrt{21}$, where upper bound could

be reached if $x=y=z=\frac{1}{3}$ $(f(\frac{1}{3},\frac{1}{3},\frac{1}{3})=3\sqrt{\frac{4}{3}+1}=\sqrt{3\cdot7}=\sqrt{21})$.

We could change the reasoning at the end. $f(x,y,z)\le\frac{7+3t}{2\sqrt{t}}$ for all $t>0$.

Hence $f(x,y,z)\le\min\limits_{t>0}\frac{7+3t}{2\sqrt{t}}=\sqrt{21}$, and since the equality in inequality

$f(x,y,z)\le\frac{7+3t}{2\sqrt{t}}$ holds for every t, then it holds and for t which minimize

the function $\frac{7+3t}{2\sqrt{t}}$ on $(0,\infty)$, if $t=\frac{7}{3}$, $\left(\frac{7+3t}{\sqrt{t}}\le\sqrt{21}\Leftrightarrow(\sqrt{3}\cdot t-\sqrt{7})^2\ge0\right)$

Reader who wants to get more confirmation that this way is effective should solve following problems and its generalization by indefinite parameter method:

Exercise 30. Find the greatest value of function

$$f(x_1,x_2,\dots,x_n)=\sqrt{cx_1+b_1}+\sqrt{cx_2+b_2}+\dots+\sqrt{cx_n+b_n}$$

with $x_1,x_2,\dots,x_n\ge0$ & $x_1+x_2+\dots+x_n=a>0$, where $c,b_1,b_2,\dots,b_n>0$

Exercise 31. Let $c,b_1,b_2,\dots,b_n$ are non-negative numbers. Find the greatest value of function:

$$f(x_1,x_2,\dots,x_n)=\sqrt[k]{(cx_1+b_1)^m}+\sqrt[k]{(cx_2+b_2)^m}+\dots+\sqrt[k]{(cx_n+b_n)^m},$$

when $x_1,x_2,\dots,x_n\ge0$, $x_1+x_2+\dots+x_n=a>0$ & $k>m$ - natural numbers.

(Hint. Use Cauchy's inequality for k numbers).

Success of that pushes me to think that in my hands is very effective way for proof of inequalities:

$$\left(\frac{a_1^q+a_2^q+\dots+a_n^q}{n}\right)^{1/q}\le\left(\frac{a_1^p+a_2^p+\dots+a_n^p}{n}\right)^{1/p}\quad \text{для } p>q \qquad (23)$$

i.e. the proof that degree average $G_p(a_1,a_2,\dots,a_n)$ is increasing function of variable $p>0$.

The proof of inequality (23) is based on inequality (19) which is written like this:

$$x^{\alpha}\cdot x^{1-\alpha}\le\alpha\cdot x+(1-\alpha)\cdot y,\ \text{for } 0<\alpha<1\ \&\ x,y>0 \qquad (24)$$

This is another way of writing the Cauchy's inequality (8) in the case when $a_1=a_2=\dots=a_k=x$ & $a_{k+1}=\dots=a_n=y$.

Actually, in this case Cauchy's inequality could be represented

$$\sqrt[n]{x^k\cdot y^{n-k}} \le \frac{k\cdot x + (n-k)\cdot y}{n}, \quad k<n \tag{25}$$

Noting $\alpha=\frac{k}{n}$, we get $\frac{n-k}{n} = 1-\alpha$ & inequality gets view of (24), where α- is rational number. Hence, reader may take inequality (23) & (24) as proved only for rational p & q.

Now the time for proving inequality (23).

Let $a_1, a_2, \ldots, a_n$, α, $t \in \mathbb{R}$, with $a_1, a_2, \ldots, a_n \ge 0$, $0<\alpha<1$. Denote $a_1+a_2+\ldots+a_n$ by M. Then for arbitrary positive number t holds inequality

$$a_i^{\alpha}\cdot t^{1-\alpha} \le \alpha\cdot a_i + (1-\alpha)\cdot t, \quad i=1,2,\ldots,n.$$

Their addition yields

$$t^{1-\alpha}\cdot(a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha}) \le \alpha\cdot M + (1-\alpha)\cdot tn$$

or it is equivalent to

$$a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha} \le (\alpha\cdot M+(1-\alpha)\cdot tn)\cdot t^{\alpha-1} \tag{26}.$$

With the equality holds when $a_i=t$, $i=1,2,\ldots,n$, i.e. accounted that $a_1+a_2+\ldots+a_n=M$, where $a_1=a_2=\ldots=a_n=t=\frac{M}{n}$.

By setting $t=\frac{M}{n}$ into inequality (26) we get

$$a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha} \le M\cdot\left(\frac{M}{n}\right)^{\alpha-1} \Leftrightarrow \frac{a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha}}{n} \le \left(\frac{M}{n}\right)^{\alpha} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha}}{n}\right)^{1/\alpha} \le \frac{a_1+\ldots a_n}{n}$$

equality occur only if $a_1=\ldots=a_n$.

Let p & q be an arbitrary positive real numbers with $q<p$.

Substitution $\alpha=\frac{q}{p}$ & $a_i=b_i^p$, $i=1,2,\ldots,n$, (where $b_1,b_2,\ldots,b_n$ are arbitrary non-negative real numbers) into inequality yields:

$$\left(\frac{a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha}}{n}\right)^{1/\alpha} \le \frac{a_1+\ldots+a_n}{n} \Leftrightarrow \left(\frac{b_1^q+\ldots+b_n^q}{n}\right)^{p/q} \le \frac{b_1^p+\ldots+b_n^p}{n}$$

$$\Leftrightarrow \left(\frac{b_1^q+b_2^q+\ldots+b_n^q}{n}\right)^{1/q} \le \left(\frac{b_1^p+b_2^p+\ldots+b_n^p}{n}\right)^{1/p}.$$

Equality occur if $b_1=b_2=\ldots=b_n$.

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq \frac{2\cdot(\alpha-1)}{\alpha^2} \qquad (28)$$

we could reason otherwise. Since inequalities in (28) hold if and only if $\alpha\cdot a=a+b$, $\alpha\cdot b=b+c$ & $\alpha\cdot c=c+a$, then addition those inequality yields $\alpha\cdot(a+b+c)=2\cdot(a+b+c)$ (where $a+b+c=1$) $\Rightarrow \alpha=2$. Since inequality (28) holds for any α, then when $\alpha=2$ holds following inequality:

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq 2\cdot\frac{2-1}{4} = \frac{1}{2}$$

with equality occur if $2a=a+b$, $2b=b+c$, $2c=c+a$, i.e. when $a=b=c=\frac{1}{3}$.
As an exercise we are going to offer you follows problems.

Exercise 32. Find a minimum of function $\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x}$ with $x,y,z>0$ & $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = 1$.

Exercise 33.
a) Let $x_1,x_2,\ldots,x_n > 0$ & $x_1+x_2+\ldots+x_n = 1$ & $k<n$. Find a minimum of of function:

$$S(x_1,x_2,\ldots,x_n) = \frac{x_1^2}{x_1+x_2+\ldots+x_k} + \frac{x_2^2}{x_2+x_3+\ldots+x_{k+1}} + \ldots + \frac{x_n^2}{x_n+x_1+\ldots+x_{k-1}}$$

b) Let $x_1,x_2,\ldots,x_n>0$ & $a_1,a_2,\ldots,a_k>0$, $k\leq n$. Prove inequality:

$$\frac{x_1^2}{a_1x_1+\ldots+a_kx_k} + \frac{x_2^2}{a_1x_2+\ldots+a_kx_{k+1}} + \ldots + \frac{x_n^2}{a_1x_n+\ldots+a_kx_{k-1}} \geq \frac{x_1+x_2+\ldots+x_n}{a_1+a_2+\ldots+a_k}$$

Example 9. Find the greatest value of function $x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}$, where $x_1+x_2+\ldots+x_n=1$ & $x_1,x_2,\ldots,x_n>0$.

Solution:
Denote $k_1+k_2+\ldots+k_n$ by m. Notice that from Cauchy's inequality follows

$$\sqrt[m]{x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}} \leq \frac{k_1x_1+k_2x_2+\ldots+k_nx_n}{m}$$

Very attracting to discuss the greatest value of $k_1x_1+k_2x_2+\ldots k_nx_n$.
But the values of $x_1,\ldots,x_n$ such which occur inequality should satisfy condition $x_1=x_2=\ldots=x_n$ and it follows that, $x_1=x_2=\ldots=x_n=\frac{1}{n}$

The greatest value of $k_1x_1+\ldots+k_nx_n$ for $x_1+x_2+\ldots+x_n=1$ & $x_1,x_2,\ldots,x_n\geq0$ could be reached if $x_1=x_2=\ldots=x_n$ only if $k_1=k_2=\ldots=k_n$. Otherwise,

function $f(x_1,x_2,\ldots,x_n)=k_1x_1+\ldots+k_nx_n$ may have values more than

Since

$$G_{-p}(b_1,b_2,\dots,b_n) = \left(\frac{b_1^{-p}+b_2^{-p}+\dots+b_n^{-p}}{n}\right)^{-1/p} = \frac{1}{\left(\frac{(b_1^{-1})^p+\dots+(b_n^{-1})^p}{n}\right)} =$$

$$= \frac{1}{G_p(b_1^{-1},\dots,b_n^{-1})} \quad \text{then}$$

first $G_{-p} \le G_o$ for $p>0$, and secondly from $o<q<p$ it follows that

$$G_q(b_1^{-1},\dots,b_n^{-1}) \le G_p(b_1^{-1},\dots,b_n^{-1}) \iff G_{-p}(b_1,b_2,\dots,b_n) \le G_{-q}(b_1,b_2,\dots,b_n)$$

Thus for any two p & $q \in \mathbb{R}$ & $b_1,b_2,\dots,b_n \ge 0$ holds (23).

It seems to me that this brief proof which is not requiring of any means for rational exponent except Cauchy's inequality is the best confirmation that it's effective to use indefinite parameters. It would be wrong to think that the possibilities of using indefinite parameter are restricting by shown above examples. We are going to show more problems in which solution the indefinite parameter altogether with other base inequalities work very effective.

Example 8. For $a,b,c>0$ & $a+b+c=1$ prove inequality:

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \ge \frac{1}{2}$$

Solution:

From inequality (7) follows that for any $x,y>0$ & $\alpha\neq 0$ holds following inequality: $\frac{(\alpha\cdot x)^2}{y} \ge 2\alpha\cdot x - y$ (equality occur only if $\alpha\cdot x=y$).

Hence, since

$$\frac{\alpha^2 a^2}{a+b} \ge 2\alpha\cdot a - (a+b), \quad \frac{\alpha^2 b^2}{b+c} \ge 2\alpha\cdot b - (b+c), \quad \frac{\alpha^2 c^2}{c+a} \ge 2\alpha\cdot c - (c+a)$$

we get

$$\alpha^2\cdot\left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right) \ge 2\alpha\cdot(a+b+c) - 2\cdot(a+b+c) = 2\alpha - 2.$$

Thus, $\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \ge \frac{2\cdot(\alpha-1)}{\alpha^2}$ for any $\alpha\neq 0$.

But then

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \ge 2\cdot\max\frac{\alpha-1}{\alpha^2}. \quad \text{Since } \frac{\alpha-1}{\alpha^2} \le \frac{1}{4} \iff (\alpha-2)^2 \ge 0,$$

then $\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \ge \frac{1}{2}$,

with the equality occur only if $\alpha\cdot a=a+b$, $\alpha\cdot b=b+c$, $\alpha\cdot c=c+a$ & $\alpha=2$, i.e. when $a=b=c=1/3$. At the moment we derived inequality:

$$f(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})=\frac{k_1+k_2+\ldots+k_n}{n}<\max\{k_1,k_2,\ldots,k_n\}.$$

If i such that $k_i=\max\{k_1,k_2,\ldots,k_n\}$, that $k_i=f(0,0,0,\ldots,1,0,\ldots0)$ (1 placed on i-th place). Therefore, it is not the way to get needed result. However, we can save it by introducing indefinite parameters $t_1,t_2,\ldots,t_n>0$. Then:

$$((t_1x_1)^{k_1}\cdot(t_2x_2)^{k_2}\cdot\ldots\cdot(t_nx_n)^{k_n})^{1/m}\le\frac{k_1t_1x_1+\ldots+k_nt_nx_n}{m}\iff$$

$$\iff x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}\le\left(\frac{k_1t_1x_1+\ldots+k_nt_nx_n}{m}\right)^m\cdot\frac{1}{t_1^{k_1}\cdot t_2^{k_2}\cdot\ldots\cdot t_n^{k_n}}.$$

With equality occur if $t_1x_1=t_2x_2=\ldots=t_nx_n$. From other hand, setting $t_i=\frac{1}{k_i}$, $i=1,2,\ldots,n$ yields for sum $k_1t_1x_1+\ldots+k_nt_nx_n$ constant value which is equal to one, that abolish the problem of concordance the equality condition with the maximum of sum $k_1t_1x_1+\ldots+k_nt_nx_n$. Hence, $\frac{x_1}{k_1}=\frac{x_2}{k_2}=\ldots=\frac{x_n}{k_n}=k$, where k is a factor of proportionality. Substituting $x_i=k\cdot k_i$, $(i=1,2,\ldots,n)$ with $x_1+x_2+\ldots+x_n=1$, we get $k=\frac{1}{k_1+\ldots+k_m}=\frac{1}{m}$. Hence, $x_i=\frac{k_i}{m}$, $i=1,2,\ldots,m$. Thus, when $t_1=\frac{1}{k_1}$, $t_2=\frac{1}{k_2}$, $\ldots$, $t_n=\frac{1}{k_n}$ the inequality could be written $x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}\le\frac{k_1^{k_1}\cdot k_2^{k_2}\cdot\ldots\cdot k_n^{k_n}}{m^m}$, where the upper bound of function $x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}$ could be reached if $x_i=\frac{k_i}{m}$, $i=1,2,\ldots,n$.

It follows that the greatest value of function $x_1^{k_1}\cdot x_2^{k_2}\cdot\ldots\cdot x_n^{k_n}$ with $x_1+x_2+\ldots+x_n=1$ & $x_1,x_2,\ldots,x_n>0$ is equal to $\frac{k_1^{k_1}\cdot k_2^{k_2}\cdot\ldots\cdot k_n^{k_n}}{(k_1+\ldots+k_n)^{k_1+\ldots+k_n}}$.

Example 10. Let $x_1,x_2,\ldots,x_n\in[a,b]$, where $0<a<b$. Prove inequality:

$$(x_1+x_2+\ldots+x_n)\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)\le\frac{n^2\cdot(a+b)^2}{4ab}$$

Solution: For arbitrary positive number t we have

$$\sqrt{(x_1+x_2+\ldots+x_n)\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)}=\sqrt{(tx_1+\ldots+tx_n)\cdot\left(\frac{1}{tx_1}+\ldots+\frac{1}{tx_n}\right)}\le$$

$$\le \frac{(tx_1+tx_2+\ldots+tx_n)+(\frac{1}{tx_1}+\ldots+\frac{1}{tx_n})}{2} \Leftrightarrow (x_1+x_2+\ldots+x_n)\cdot(\frac{1}{x_1}+\ldots+\frac{1}{x_n})\le$$

$$\le \frac{1}{4}\cdot((tx_1+\frac{1}{tx_1})+\ldots+(tx_n+\frac{1}{tx_n}))^2 \qquad (29)$$

Let $0<c<d$. Prove that for any $x\in[c,d]$ holds inequality: $x+\frac{1}{x} \le \max\{c+\frac{1}{c},d+\frac{1}{d}\}$. Notice first that $x+\frac{1}{x}\le y+\frac{1}{y} \Leftrightarrow y-x\ge\frac{y-x}{xy} \Leftrightarrow$ $\Leftrightarrow (y-x)\cdot(1-xy)\ge 0$. Let $c<x<d$. If $x+\frac{1}{x}\le c+\frac{1}{c}$, then $x+\frac{1}{x}\le \max\{c+\frac{1}{c},d+\frac{1}{d}\}$ Suppose that $x+\frac{1}{x} > c+\frac{1}{c}$. This inequality is equivalent to $(x-c)\cdot(xc-1)>0 \Leftrightarrow xc>1 \Rightarrow xd>1 \Leftrightarrow (d-x)\cdot(dx-1)>0 \Leftrightarrow x+\frac{1}{x}>d+\frac{1}{d}$.

Thus, for any $x\in[c,d]$ $x+\frac{1}{x} \le \max\{c+\frac{1}{c},d+\frac{1}{d}\}$ and since the upper bound could be reached then $\max_{x\in[a,b]}\{x+\frac{1}{x}\} = \max\{c+\frac{1}{c},d+\frac{1}{d}\}$.

Turn back to the main inequality. Since $x_i\in[a,b]$, then $tx_i\in[ta,tb]$ & $tx_i+\frac{1}{tx_i}\le \max\{ta+\frac{1}{ta},tb+\frac{1}{tb}\}$, $i=1,\ldots,n$. Notice that $ta+\frac{1}{ta}=tb+\frac{1}{tb} \Leftrightarrow$ $t^2ab-1=0 \Leftrightarrow t=\frac{1}{\sqrt{ab}}$. Since inequality (29) is true for any t, then in particular its true for $t=\frac{1}{\sqrt{ab}}$, and then $tx_i+\frac{1}{tx_i}\le \sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\frac{a+b}{\sqrt{ab}}$, it means that

$$(x_1+x_2+\ldots+x_n)\cdot(\frac{1}{x_1}+\ldots+\frac{1}{x_n}) \le \frac{1}{4}\cdot(n\cdot\frac{a+b}{\sqrt{ab}})^2 = n^2\cdot\frac{(a+b)^2}{4ab} \qquad (30)$$

When n is an even, i.e. $n=2k$ for some k and upper bound is reachable if $x_1=x_2=\ldots=x_k=a$ и $x_{k+1}=\ldots=x_n=b$, but if n is an odd number it's wrong. Searching for upper bound for any n, i.e. maximum of function

$$h(x_1,x_2,\ldots,x_n) = (x_1+x_2+\ldots+x_n)\cdot(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}) \quad \text{if } x_i\in[a,b],\ 0<a<b\ -$$

is a complicate problem and it can be solved by other ways. (See "DELTA"'s competition No.10).

We're going to show one more example of using indefinite parameter method in the proof of *Cauchy-Bunyakovsky-Shwartz's inequality* which is playing in mathematics and applications very important role. We are not going to stay on explicit geometric interpretation of this inequality connected with mesures of length, angles and with conception of distance in finite & infinite spaces. However, we shall use the notion system in which you can see well-known inner product from vector algebra.

Theorem 4. For any $x_1,x_2,\ldots,x_n$ & $y_1,y_2,\ldots,y_n$ holds following Cauchy-Bunyakovsky-Shwartz (CBS) inequality:

$$(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \ldots + x_n^2)\cdot(y_1^2 + y_2^2 + \ldots + y_n^2)$$

with equality occur if and only if there exist $k\in\mathbb{R}$ such that $x_i=ky_i$, $i=1,2,\ldots,n$. (See strict definition of ordered set on page 88)

Proof: For two ordered sets $x=(x_1,x_2,\ldots,x_n)$ & $y=(y_1,y_2,\ldots,y_n)$.
Note the sum $x_1y_1+x_2y_2+\ldots+x_ny_n$ by $S(x,y)$.

Notice following facts - simple properties of $S(x,y)$. (However, the reader should prove it by himself).

1. $S(x,x) \ge 0$ for any set x and $S(x,x)=0$ if and only if $x=0$, i.e. when $x=(0,0,\ldots,0)$.
2. $S(x,y) = S(y,x)$ for any x & y.
3. $S(kx,y) = k\cdot S(x,y)$, where k - any real number, x & y - arbitrary sets From 2) & 3) follow that $S(x,ky) = k\cdot S(x,y)$
4. $S(x+y,z) = S(x,z) + S(y,z)$, for any sets of n-numbers x, y & z.

(Here $kx=(kx_1,kx_2,\ldots,kx_n)$ & $x+y=(x_1+y_1,\ldots,x_n+y_n)$).

By those notions the CBS inequality could be rewritten as

$$S^2(x,y)\le S(x,x)\cdot S(y,y)$$

The proof is very short:
From 1) property for any sets of n numbers x & y and any number t (here is a parameter !) holds inequality $S(x-ty,x-ty) \ge 0$ when equality occur if and only if $x-ty=0$ or $x=ty$. From other hand, using properties 2),3),4) we obtain $S(x-ty,x-ty) = S(x,x) - 2t\cdot S(x,y) + t^2\cdot S(y,y)$

Considering $S(x-ty,x-ty)$ as quadratic three terms relatively parameter t, we conclude that $S(x-ty,x-ty)\ge 0$ for any t if and only if the discriminant satisfy inequality:

$$S^2(x,y) - S(x,x)\cdot S(y,y) \le 0 \iff S^2(x,y) \le S(x,x)\cdot S(y,y).$$

In this case equality $S(x-ty,x-ty)=0$ as quadratic three terms relatively variable t is equal to zero if and only if:
$S^2(x,y)=S(x,x)\cdot S(y,y)$, from property 1) it's possible only if $x-ty=0$, i.e. $x_i=ty_i$, $i=1,2,\ldots,n$. That was needed to prove.■

The sets $x=(x_1,x_2,\ldots,x_n)$ & $y=(y_1,y_2,\ldots,y_n)$ could be considered as n-dimentional vectors. Equality condition of CBS inequality can be considered as collinearity condition of vectors x & y.
Of course the CBS inequality we could get as a particular case of Gelder's inequality (exercise 26(b)) with $p=2$. However, as said before we are interesting in ways of deriving but not in result by itself. In this case the closed (by references) proof is preferable.
Consider the applications of CBS inequality to inequalities proofs and problems solutions:

1. Inequality $(a_1^2+a_2^2+\ldots+a_n^2)\cdot n \ge (a_1+a_2+\ldots+a_n)$ is simple corollary from CBS inequality when $x_i=a_i$ & $y_i=1$.
2. Setting in CBS inequality for $a_1,a_2,\ldots,a_n>0$ $x_i=\dfrac{a_i}{\sqrt{a_{i+1}}}$, $y_i=\sqrt{a_{i+1}}$,

$i=1,2,\ldots,n$ и $x_n=\dfrac{a_n}{\sqrt{a_1}}$, $y_n=\sqrt{a_1}$, we get:

$$\left(\frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\ldots+\frac{a_n^2}{a_1}\right)\cdot(a_2+a_3+\ldots+a_{n-1}+a_1)\ge(a_1+a_2+\ldots+a_n)^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{a_1^2}{a_2}+\frac{a_2^2}{a_3}+\ldots+\frac{a_n^2}{a_1}\ge a_1+a_2+\ldots+a_n,$$ when equality occur if

$\dfrac{a_1}{a_2}=\dfrac{a_2}{a_3}=\ldots=\dfrac{a_n}{a_1} \Leftrightarrow a_1=a_2=\ldots=a_n$. Similarly we can prove more general inequality which was considered in exercise 33.

3. Let $a,b,c>0$

$$\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2=\left(\frac{a}{b}\cdot\frac{b}{c}+\frac{c}{a}\cdot\frac{a}{b}+\frac{b}{c}\cdot\frac{c}{a}\right)^2\le\left(\frac{a^2}{b^2}+\frac{c^2}{a^2}+\frac{b^2}{c^2}\right)\cdot\left(\frac{b^2}{c^2}+\frac{a^2}{b^2}+\frac{c^2}{a^2}\right)$$

$$\Leftrightarrow \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

4. Prove inequality:

$$\frac{(a_1+a_2+\ldots+a_n)^2}{2\cdot(a_1^2+a_2^2+\ldots+a_n^2)}\le\frac{a_1}{a_2+a_3}+\ldots+\frac{a_n}{a_1+a_2},\quad \text{для}\quad a_1,a_2,\ldots,a_n>0$$

Proof. Since $a_i=\sqrt{\dfrac{a_i}{a_{i+1}+a_{i+2}}}\cdot\sqrt{a_i(a_{i+1}+a_{i+2})}$, $i=1,2,\ldots,n-2$,

$a_{n-1}=\sqrt{\dfrac{a_{n-1}}{a_n+a_1}}\cdot\sqrt{a_{n-1}(a_n+a_1)}$, $a_n=\sqrt{\dfrac{a_n}{a_1+a_2}}\cdot\sqrt{a_n(a_1+a_2)}$, then by CBS inequality

$$(a_1+a_2+\ldots+a_n)^2\le\left(\frac{a_1}{a_2+a_3}+\ldots+\frac{a_n}{a_1+a_2}\right)\cdot\left((a_1\cdot(a_2+a_3)+\ldots+a_n\cdot(a_1+a_2)\right)\le$$

$$\le\left(\frac{a_1}{a_2+a_3}+\ldots+\frac{a_n}{a_1+a_2}\right)\cdot\left(\left(\frac{a_1^2+a_2^2}{2}+\frac{a_1^2+a_3^2}{2}\right)+\left(\frac{a_2^2+a_3^2}{2}+\frac{a_2^2+a_4^2}{2}\right)+\ldots+\left(\frac{a_{n-1}^2+a_n^2}{2}\right.\right.$$

$$\left.+\frac{a_{n-1}^2+a_1^2}{2}\right)+\left(\frac{a_n^2+a_1^2}{2}+\frac{a_n^2+a_2^2}{2}\right)=\left(\frac{a_1}{a_2+a_3}+\ldots+\frac{a_n}{a_1+a_2}\right)\cdot(2a_1^2+2a_2^2+\ldots+2a_n^2),$$

since every element a_i appears with four others elements, i.e.

$a_{i-2}(a_{i-1}+a_i)$, $a_{i-1}(a_i+a_{i+1})$, $a_i(a_{i+1}+a_{i+2})$. ■

5. Given that $x^2+3y^2+z^2=2$. Find the smallest value of function $2x+y-z$.

Solution:

$$(2x+y-z)^2 = \left(2\cdot x + \frac{1}{\sqrt{3}}\cdot\sqrt{3}\cdot y - 1\cdot z\right)^2 \le (x^2+3y^2+z^2)\cdot\left(2^2+\left(\frac{1}{\sqrt{3}}\right)^2+1^2\right) \iff$$

$$\iff (2x+y-z)^2 \le 2\cdot\left(4+\frac{1}{3}+1\right) = \frac{32}{3}.$$ Hence, $|2x+y-z| \le 4\sqrt{\frac{2}{3}}$.

Equality condition is $\frac{x}{2} = \frac{\sqrt{3}\cdot y}{1/\sqrt{3}} = \frac{z}{-1} \iff \frac{x}{2} = 3y = -z = t$, t is a parameter.

Hence, $x=2t$, $y=\frac{t}{3}$, $z=-t$. Since $x^2+3y^2+z^2=2$, then $4t^2+\frac{t^2}{3}+t^2=2 \iff$

$\iff t^2=\frac{3}{8} \iff |t| = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}$. Since $-4\cdot\sqrt{\frac{2}{3}} \le 2x+y-z \le 4\cdot\sqrt{\frac{2}{3}}$, then when

$t=\frac{1}{2}\sqrt{\frac{3}{2}}$ & $x=\sqrt{\frac{2}{3}}$, $y=\frac{1}{2\sqrt{6}}$, $z=-\frac{1}{2}\sqrt{\frac{3}{2}}$, $2x+y-z=\sqrt{6}+\frac{1}{2\sqrt{6}}+\frac{3}{2\sqrt{6}}=\frac{16}{2\sqrt{6}}=4\sqrt{\frac{2}{3}}$

we reach the greatest value of function $2x+y-z$, and when

$t=-\frac{1}{2}\sqrt{\frac{3}{2}}$, $x=-\sqrt{\frac{3}{2}}$, $y=-\frac{1}{2\sqrt{6}}$, $z=\frac{1}{2}\sqrt{\frac{3}{2}}$ reach the smallest value of function $2x+y-z$.

6. Solve following system

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y + 1990\cdot z = 1991. \end{cases}$$

Solution. Suppose x, y, z is solution of system then

$1991^2 \le (x\cdot 1+y\cdot 1+z\cdot 1990)^2 \le (1^2+1^2+1990^2)\cdot(x^2+y^2+z^2) \iff 1991^2 \le 1990^2+2 \iff$

$\iff 2\cdot 1990+1 \le 2$. We got contradiction. The system has no solution.

Exercise 34. Find greatest and smallest values of function

$a_1x_1 + a_2x_2 + a_3x_2$, with $b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = c$, $c, b_1, b_2, b_3 \ge 0$.

Exercise 35. Find greatest value of function $x + y + z$ when $x^2+2y^2+z^2+xy-xz-yz = 1$.

Exercise 36. Solve system:

$$\begin{cases} x_1 + x_2 + \ldots + x_n = 1 \\ x_1^2 + x_2^2 + \ldots + x_n^2 = \frac{1}{n} \end{cases}$$

Exercise 37. Prove inequality:

$$(x+y+z)^2 + (x+a)^2 + (y+b)^2 + (z+c)^2 \ge \frac{1}{4}\cdot(a+b+c)^2$$

Find out when equality occur.

Exercise 38. Prove inequality:

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}(x_i+a)^2 \ge \frac{n}{n+1}\cdot a^2$$

When does equality occur ?

Variation 4.

Here we go back to the problems and ideas from variation 1 which is linked with concordant pairs and further development of ideas on high technic level. As before we'll start from concrete inequalities.
We remind you the definition of concordance pairs. Two ordered pairs of numbers (a,b) & (c,d) are concordant in order if $(a-b)\cdot(c-d)\ge 0$.

Example 11. Prove inequality:

$$x^x\cdot y^y\cdot z^z \ge (xyz)^{(x+y+z)/3}$$

Solution.
Pairs (x,y) & (ln x, ln y) are concordant in order since f(t)=ln t is monotone increasing onto domain of definition, it means

$$(x-y)\cdot(\ln x - \ln y)\ge 0 \Leftrightarrow x\cdot \ln x + y\cdot \ln y \ge x\cdot \ln y + y\cdot \ln x \Leftrightarrow e^{x\ln x+y\ln y} \ge$$

$$\ge e^{x\ln y+y\ln x} \Leftrightarrow x^x\cdot y^y \ge y^x\cdot x^y \Leftrightarrow x^x\cdot y^y \ge (xy)^{(x+y)/2}. \quad (x,y>0).$$

The same result we could get without involving of logarithms, namely on the base of following thing. If pairs (a,b) & (c,d) are concordant in order and numbers a & b are positive, then $(\frac{a}{b})^{c-d} \ge 1$.

Actually, if $a\ge b$ then $\frac{a}{b}\ge 1$ and therefore $c\ge d$ it means $(\frac{a}{b})^{c-d}\ge 1$.

If $a\le b$ then by considered case $(\frac{b}{a})^{d-c}\ge 1 \Leftrightarrow (\frac{a}{b})^{c-d}\ge 1$.

Rewrite last inequality so $a^c\cdot b^d \ge b^c\cdot a^d$.

Thus, we've got that for concordant in order pairs of numbers (a,b) & (c,d), where a,b>0 holds inequality $a^c\cdot b^d\ge b^c\cdot a^d$.

In particular for any x,y>0 pairs (x,y) & (x,y) are concordant in order, hence $x^x\cdot y^y\ge x^y\cdot y^x$. Multiply both of sides by $x^x\cdot y^y$ and extracting the square root yields $x^x\cdot y^y\ge (xy)^{(x+y)/2}$.

Multiply following inequality $(\frac{x}{y})^{x-y}\ge 1$, $(\frac{y}{z})^{y-z}\ge 1$, $(\frac{z}{x})^{z-x}\ge 1$, we get:

$$(\frac{x}{y})^{x-y}\cdot(\frac{y}{z})^{y-z}\cdot(\frac{z}{x})^{z-x} \ge 1 \Leftrightarrow \frac{x^x\cdot y^y}{y^x\cdot x^y}\cdot\frac{y^y\cdot z^z}{y^z\cdot z^y}\cdot\frac{z^z\cdot x^x}{z^x\cdot x^z} \ge 1 \Leftrightarrow$$

$$x^{2x}y^{2y}z^{2z} \ge x^{y+z}y^{x+z}z^{x+y} \Leftrightarrow x^{3x}y^{3y}z^{3z} \ge (xyz)^{x+y+z} \Leftrightarrow x^xy^yz^z \ge (xyz)^{(x+y+z)/3}$$

Now appear a suggestion that holds following inequality

$$x_1^{x_1}\cdot x_2^{x_2}\cdot\ldots\cdot x_n^{x_n} \ge (x_1x_2\ldots x_n)^{\frac{x_1+x_2+\ldots+x_n}{n}}, \quad x_1,x_2,\ldots,x_n>0$$

Taking logarithms of this inequality yields:

$$\frac{x_1\cdot\ln x_1 + x_2\cdot\ln x_2 + \ldots + x_n\cdot\ln x_n}{x_1+x_2+\ldots+x_n} \ge \frac{\ln x_1 + \ln x_2 + \ldots + \ln x_n}{n} \quad (31)$$

This representation of inequality pushes on an idea that if inequality is true then the cause of that is concordance of pairs (x_i, x_j) & $(\ln x_i, \ln x_j)$, which follow from monotone increasing of logarithm function. Appear a suggestion that this inequality will be true if instead of ln x we take an arbitrary monotone function, i.e. we talk about inequality:

$$\frac{x_1 \cdot f(x_1) + x_2 \cdot f(x_2) + \ldots + x_n \cdot f(x_n)}{x_1 + x_2 + \ldots + x_n} \geq \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \qquad (32)$$

with $x_1, x_2, \ldots, x_n$ belong to domain where function f(x) is monotone increasing (precisely, decreasing).

For the proof we need two identities:

1. $\sum_{1\le i<j\le n} (a_i + a_j) = (n-1)\cdot\sum_{i=1}^{n} a_i$

2. $\sum_{1\le i<j\le n} (a_i b_j + a_j b_i) = \left(\sum_{i=1}^{n} a_i\right)\cdot\left(\sum_{i=1}^{n} b_i\right) - \sum_{i=1}^{n} a_i b_i$

Proof:

1. $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i+a_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} 1 + \sum_{j=1}^{n} a_j \sum_{i=1}^{n} 1 = n\cdot\sum_{i=1}^{n} a_i +$ $+ n\cdot\sum_{j=1}^{n} a_j = 2n\cdot\sum_{i=1}^{n} a_i$. From other hand, $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i+a_j) = \sum_{i=1}^{n}(a_i+a_i) + \sum_{1\le i<j\le n}(a_i+a_j)$ $+ \sum_{1\le i<j\le n}(a_i+a_j) = 2\cdot\sum_{i=1}^{n} a_i + 2\cdot\sum_{1\le i<j\le n}(a_i+a_j)$, since $\sum_{1\le i<j\le n}(a_i+a_j) = \sum_{1\le j<i\le n}(a_i+a_j)$

It means $2n\cdot\sum_{i=1}^{n} a_i = 2\cdot\sum_{1\le i<j\le n}(a_i+a_j) + 2\cdot\sum_{i=1}^{n} a_i \iff \sum_{1\le i<j\le n}(a_i+a_j) = (n-1)\cdot\sum_{i=1}^{n} a_i$.

2. $\left(\sum_{i=1}^{n} a_i\right)\cdot\left(\sum_{j=1}^{n} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j = \sum_{i=1}^{n} a_i b_i + \sum_{1\le i<j\le n} a_i b_j + \sum_{1\le j<i\le n} a_i b_j = \sum_{i=1}^{n} a_i b_i +$ $+ \sum_{1\le i<j\le n}(a_i b_j + a_j b_i)$, since $\sum_{1\le j<i\le n} a_i b_j = \sum_{1\le i<j\le n} a_j b_i$.

Now we can take to prove the inequality. Let f is monotone non-decreasing onto domain D & $x_1, x_2, \ldots, x_n \in D$. Since pairs (x_i, y_i) & $(f(x_i), f(x_j))$ are concordant then

$$0 \le \sum_{1\le i<j\le n}(x_i - x_j)\cdot(f(x_i) - f(x_j)) = \sum_{1\le i<j\le n}(x_i f(x_i) + x_j f(x_j)) - \sum_{1\le i<j\le n}(x_i f(x_j) + x_j f(x_i))$$

$$= (n-1)\cdot\sum_{i=1}^{n} x_i f(x_i) - \left(\left(\sum_{i=1}^{n} x_i\right)\cdot\left(\sum_{j=1}^{n} f(x_j)\right) - \sum_{i=1}^{n} x_i f(x_i)\right) = n\cdot\sum_{i=1}^{n} x_i f(x_i) -$$

$$-\left(\sum_{i=1}^{n} x_i\right)\cdot\left(\sum_{i=1}^{n} f(x_i)\right) \quad\Leftrightarrow\quad n\cdot\sum_{i=1}^{n} x_i f(x_i) \geq \left(\sum_{i=1}^{n} x_i\right)\cdot\left(\sum_{i=1}^{n} f(x_i)\right) \qquad (33)$$

If $x_1, x_2, \ldots, x_n > 0$, then inequality we can rewrite in the view of (32). Substitution $f(x)=\ln x$ into (33) yields (31).

Exercise 39. a) Prove inequality:

$$3\cdot(a^3+b^3+c^3) \leq (a+b+c)\cdot(a^2+b^2+c^2), \qquad a,b,c\geq 0$$

b) Let f(x) is a monotone non-decreasing function. Prove inequality:

$$\frac{f(1) + 2\cdot f(2) + \ldots + n\cdot F(n)}{n+1} \geq \frac{f(1) + f(2) + \ldots + f(n)}{2}$$

Go back to inequality $a_1b_1+a_2b_2 \geq a_1b_2+a_2b_1$, where ordered pairs (a_1, a_2) & (b_1, b_2) are concordant in order and we are going to try generalize for greater number of variables. For the beginning consider case $n=3$. Then for any $1\leq i<j\leq 3$, $1\leq k<m\leq 3$ pairs (a_i, a_j) & (b_k, b_m) are oncordant in order and it means $a_ib_k+a_jb_m \geq a_ib_m+a_jb_k$. Hence we get following chains of inequalities (over and under arrows pointed concordant pairs).

Suppose that $a_1\leq a_2\leq a_3$ & $b_1\leq b_2\leq b_3$, then

1. $a_1b_1+a_2b_2+a_3b_3 \geq a_1b_1+a_2b_3+a_3b_2 \geq a_1b_3+a_2b_1+a_3b_2 \geq a_1b_3+a_2b_2+a_3b_1$

(over the first step: (a_2, a_3) и (b_2, b_3); under the second step: (a_1, a_2) и (b_1, b_3); over the third step: (a_2, a_3) и (b_1, b_3))

2. $a_1b_1+a_2b_2+a_3b_3 \geq a_1b_2+a_2b_1+a_3b_3 \geq a_1b_2+a_2b_3+a_3b_1 \geq a_1b_3+a_2b_2+a_3b_1$

(over the first step: (a_1, a_2) и (b_1, b_2); under the second step: (a_2, a_3) и (b_1, b_3); over the third step: (a_1, a_2) и (b_2, b_3))

Transformations in chains correspond to the following chain of permutations of numbers (1,2,3):

1. (1,2,3) → (1,3,2) → (3,1,2) → (3,2,1)
2. (1,2,3) → (2,1,3) → (2,3,1) → (3,2,1)

Number of different permutations (1,2,3) is equal to six: (1,2,3),(1,3,2),(3,1,2),(2,1,3),(3,2,1).

From all we've done we can make conclusion which is completely included in following inequality:

$$a_1b_3+ a_2b_2+ a_3b_1 \leq a_1b_{i_1} + a_2b_{i_2} + a_3b_{i_3} \leq a_1b_1+ a_2b_2+ a_3b_3 \qquad (34)$$

where $a_1\leq a_2\leq a_3$, $b_1\leq b_2\leq b_3$; (i_1, i_2, i_3) is an arbitrary permutation of numbers (1,2,3).

In what follows we shall say that ordered triples (a_1,a_2,a_3) & (b_1,b_2,b_3) are concordant in order if for any $1\le i<j\le 3$ the pairs (a_i,a_j) & (b_i,b_j) are concordant in order.

Suppose that triples (a_1,a_2,a_3) & (b_1,b_2,b_3) are concordant in order. Then there exist a permutation (j_1,j_2,j_3) of numbers $(1,2,3)$ such that $a_{j_1}\le a_{j_2}\le a_{j_3}$. But then $b_{j_1}\le b_{j_2}\le b_{j_3}$. Denote $x_k=a_{j_k}$, $y_k=b_{j_k}$, $k=1,2,3$.

Let (i_1,i_2,i_3) is an arbitrary permutation of numbers $(1,2,3)$. Consider the sum

$$a_1b_{i_1}+a_2b_{i_2}+a_3b_{i_3} = a_{j_1}b_{i_{j_1}}+a_{j_2}b_{i_{j_2}}+a_{j_3}b_{i_{j_3}}.$$

There exist a permutation (k_1,k_2,k_3) of numbers $(1,2,3)$ such that $i_{j_1}=j_{k_1}$, $i_{j_2}=j_{k_2}$, $i_{j_3}=j_{k_3}$. Actually, i_{j_1} is one of the numbers $1,2,3$ and therefore in ordered set (j_1,j_2,j_3) its placed on k_1-th place for some k_1, i.e. $j_{k_1}=i_{j_1}$. Similarly, numbers i_{j_2} & i_{j_3} keep places k_2 & k_3. (Each number keep only one place). Then:

$$a_1b_1+a_2b_2+a_3b_3 = a_{j_1}b_{j_1}+a_{j_2}b_{j_2}+a_{j_3}b_{j_3} = x_1y_1+x_2y_2+x_3y_3 \ge x_1y_{k_1}+x_2y_{k_2}+x_3y_{k_3}$$

$$= a_{j_1}b_{j_{k_1}}+a_{j_2}b_{j_{k_2}}+a_{j_3}b_{j_{k_3}} = a_{j_1}b_{i_{j_1}}+a_{j_2}b_{i_{j_2}}+a_{j_3}b_{i_{j_3}} = a_1b_{i_1}+a_2b_{i_2}+a_3b_{i_3},$$

as long as numbers x_1,x_2,x_3 & y_1,y_2,y_3 can be set to inequality (34).

Thus for any two ordered triples (a_1,a_2,a_3) & (b_1,b_2,b_3) which are concordant in order, holds inequality for any permutation:

$$a_1b_3+a_2b_2+a_3b_1 \le a_1b_{i_1}+a_2b_{i_2}+a_3b_{i_3} \le a_1b_1+a_2b_2+a_3b_3 \qquad (35)$$

since sets (a_1,a_2,a_3) & $(-b_3,-b_2,-b_1)$ are concordant then

$$a_1(-b_3)+a_2(-b_2)+a_3(-b_1) \ge a_1(-b_{i_1})+a_2(-b_{i_2})+a_3(-b_{i_3}).$$

Inequality (35) gives a chance to prove inequalities which are difficult indeed. To be convinced of that sufficiently to prove inequalities which will be written below without references to inequality (35).

Let $a,b,c>0$. Prove inequalities:

1. $\dfrac{a^4}{c}+\dfrac{b^4}{a}+\dfrac{c^4}{b}\ge a^3+b^3+c^3.$

Solution. Triples (a^4,b^4,c^4) & $\left(-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}\right)$ are concordant, then

$$a^4\left(-\frac{1}{a}\right)+b^4\left(-\frac{1}{b}\right)+c^4\left(-\frac{1}{c}\right)\ge a^4\left(-\frac{1}{c}\right)+b^4\left(-\frac{1}{a}\right)+c^4\left(-\frac{1}{b}\right) \iff$$

$$\Leftrightarrow\quad a^3+b^3+c^3 \le \frac{a^4}{c}+\frac{b^4}{a}+\frac{c^4}{b}.$$

2. $$\frac{a^3}{b^2+bc+c^2}+\frac{b^3}{c^2+ca+a^2}+\frac{c^3}{a^2+ab+b^2} \ge \frac{a+b+c}{3}$$

Solution. Triples (a^3,b^3,c^3) & $\left(\frac{1}{b^2+bc+c^2}, \frac{1}{c^2+ca+a^2}, \frac{1}{a^2+ab+b^2}\right)$ are concordant. Really

$$(a^3-b^3)\left(\frac{1}{b^2+bc+c^2}-\frac{1}{c^2+ca+a^2}\right)=\frac{(a^3-b^3)\cdot(c^2+ca+a^2-b^2-bc-c^2)}{(b^2+bc+c^2)\cdot(c^2+ca+a^2)}=\frac{(a^3-b^3)(a-b)(a+b+c)}{(b^2+bc+c^2)(c^2+ca+a^2)}$$

Similarly, can be proved the concordance of other triples.

Hence,

$$\frac{a^3}{b^2+bc+c^2}+\frac{b^3}{c^2+ca+a^2}+\frac{c^3}{a^2+ab+b^2} \ge \frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^3+bc+c^2}+\frac{c^3}{c^2+ca+a^2}.$$

Required inequality is following from inequality $\frac{3x^3}{x^2+xy+y^2} \ge 2x - y.$

3. $$\frac{a^5}{b^2c^2}+\frac{b^4}{a^2c^2}+\frac{c^4}{a^2b^2} \ge a+b+c.$$

Solution:

Triples (a^5,b^5,c^5) & $\left(\frac{1}{b^2c^2}, \frac{1}{c^2a^2}, \frac{1}{a^2b^2}\right)$ are concordant, hence

$$a^5\cdot\frac{1}{b^2c^2}+b^5\cdot\frac{1}{c^2a^2}+c^5\cdot\frac{1}{a^2b^2} \ge \frac{a^5}{c^2a^2}+\frac{b^5}{a^2b^2}+\frac{c^5}{b^2c^2}=\frac{a^3}{c^2}+\frac{b^3}{a^2}+\frac{c^3}{b^2}.$$

Triples (a^3,b^3,c^3) & $\left(-\frac{1}{a^2},-\frac{1}{b^2},-\frac{1}{c^2}\right)$ are concordant, therefore

$$a^3\cdot\left(-\frac{1}{a^2}\right)+b^3\cdot\left(-\frac{1}{b^2}\right)+c^3\cdot\left(-\frac{1}{c^2}\right) \ge a^3\cdot\left(-\frac{1}{c^2}\right)+b^3\cdot\left(-\frac{1}{a^2}\right)+c^3\cdot\left(-\frac{1}{b^2}\right)$$

$$\Leftrightarrow\quad a^2+b^2+c^2 \le \frac{a^3}{c^2}+\frac{b^3}{a^2}+\frac{c^3}{b^2}.$$

Thus, $$\frac{a^5}{b^2c^2}+\frac{b^5}{a^2c^2}+\frac{c^5}{a^2b^2} \ge a+b+c.$$

From this directly follows inequality

$$a^7+b^7+c^7 \ge a^2b^2c^3+a^3b^2c^2+a^2b^3c^2,\ \text{для}\ a,b,c\ge 0$$

Exercise 40. Prove following inequalities by finding concordant triples out

a) inequality from example 5;

b) inequality from exercise 9(b);

c) $a^7b + b^7c + c^7a \geq a^2b^2c^2\cdot(ab+bc+ca)$, $a,b,c\geq 0$

d) $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$, $a,b,c>0$.

And two more problems. (IMO XXIV):
4. Let a,b,c be length of sides. Prove inequality:

$$a^2b\cdot(a-b) + b^2c\cdot(b-c) + c^2a\cdot(c-a) \geq 0 \qquad (36)$$

Notice that if we denote $a+b-c = z$, $b+c-a = x$, $c+a-b = z$, then we get an expression for a,b,c through independent positive variables x,y,z:

$a=\frac{y+z}{2}$, $b=\frac{x+z}{2}$, $c=\frac{x+y}{2}$. As result of substitution a,b,c into inequality (36) and complicate algebraic transformations we get:
$x^3z + y^3z + z^3y \geq x^2yz + y^2xz + z^2xy$, with which we have already met (in exercise 33 and example 7).

From other hand, concordant triples give a chance for immediately proof of inequality (36). Remove the brackets and rewrite (36) as

$$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Addition to the both of sides sum $a^2bc + ab^2c + abc^2$ yields:

$$ab\cdot(a^2+bc) + bc\cdot(b^2+ac) + ca\cdot(c^2+ab) \geq ab\cdot(c^2+ab) + bc\cdot(a^2+bc) + ac\cdot(b^2+ac)$$

Now we're going to prove that triples (a^2+b, b^2+ac, c^2+ab) и $(-bc,-ac,-ab)$ are concordant.
Actually,

$$(a^2+bc-b^2-ac)\cdot(ac-bc) = c\cdot(a-b)\cdot(a-b)\cdot(a+b-c) = c\cdot(a-b)^2\cdot(a+b-c) > 0$$

$$(b^2+ac-c^2-ab)\cdot(ab-ac) = a\cdot(b-c)^2\cdot(b+c-a) > 0$$

$$(a^2+bc-c^2-ab)\cdot(ab-bc) = b\cdot(a-c)^2\cdot(a+c-b) > 0.$$

From concordance of pointed triples follows that:

$$-bc\cdot(a^2+bc)-ac\cdot(b^2+ac)-ab\cdot(c^2+ab) \geq -ab\cdot(a^2+bc)-bc\cdot(b^2+ac)-ca\cdot(c^2+ab) \iff$$

$$\iff ab\cdot(a^2+bc)+bc\cdot(b^2+ac)+ca\cdot(c^2+ab) \geq bc\cdot(a^2+bc)+ac\cdot(b^2+ac)+ab\cdot(c^2+ab).$$

5. Let a, b, c sides of arbitrary triangle. Prove inequality

$$2\cdot\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge \frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3$$

Solution. Rewrite initial inequality :

$$c\cdot(a^2+bc) + a\cdot(b^2+ac) + b\cdot(c^2+ab) \ge a^2b + b^2c + c^2a + 3abc$$

Triples (a,b,c) & (a^2,b^2,c^2) are concordant. Therefore,

$$a^3+b^3+c^3 \ge a^2b+b^2c+c^2a \iff a^3+b^3+c^3+3abc \ge a^2b+b^2c+c^2a+3abc$$

From other hand, triples (a,b,c) & $(-a^2-bc,-b^2-ac,-c^2-ab)$ are concordant it follows that

$$a\cdot(b^2+ac) + b\cdot(c^2+ab) + c\cdot(a^2+bc) \ge a\cdot(a^2+bc) + b\cdot(b^2+ac) + c\cdot(c^2+ab) =$$

$$= a^3 + b^3 + c^3 + 3abc \iff a\cdot(b^2+ac) + b\cdot(c^2+ab) + c\cdot(a^2+bc) \ge a^3+b^3+c^3+3abc$$

$$\ge a^2b + b^2c + c^2a + 3abc.$$

Convincing arguments of efficiency of using inequality (35) and desire to generalize it by case of arbitrary n, become sufficiently motivated. However, before carrying that out we have to make some preparation to simplify the wording and proof.

Definition. Any function defined onto set $\{1,2,\dots,n\}$ we shall call *ordered set of* n *numbers*. If every $i\in\{1,2,\dots,n\}$ correspond to x_i, then ordered set used to note $x=(x_1,x_2,\dots,x_n)$, i.e. on the i-th place we have the x_i and also each of elements (value of function) is strictly reserved by its place (It's a value of argument in the order which is defined by consequent natural numbers).

From definition follows $(x_1,x_2,\dots,x_n)=(y_1,y_2,\dots,y_n) \iff x_i=y_i,\ i=1,2,\dots,n.$

Number x_i is to be said i-th component (i-th coordinate of set x).

Definition. We shall call by *permutation* the ordered set of n numbers taken one by one from the set $\{1,2,\dots,n\}$.
For the notion of permutation we shall use Greek letters $\alpha,\beta,\gamma,\delta,\dots$
Then sentence: "α is a permutation of $\{1,2,\dots,n\}$" means that on the i-th place of permutation placed number $\alpha(i)\in\{1,2,\dots,n\}$, with $\alpha(i)\ne\alpha(j)$ if $i\ne j$. Clearly that $\{\alpha(1),\alpha(2),\dots,\alpha(n)\}=\{1,2,\dots,n\}$, but $(\alpha(1),\alpha(2),\dots,\alpha(n)) = (1,2,\dots,n)$ only if $\alpha(i)=i$, $i=1,2,\dots,n$, because two ordered sets of n numbers are equal if and only if elements of sets placed on the same place are equal. And ordered set $(1,2,\dots,n)$ we shall denote by ε, $\varepsilon(i)=i$, $i=1,2,\dots,n$.

Example: $\alpha=(5,3,2,4,1)$ - permutation of set $\{1,2,\dots,5\}$ (or we're saying permutation of numbers 1,2,3,4,5.) $\alpha(1)=5$, $\alpha(2)=3$, $\alpha(3)=2$, $\alpha(4)=4$, $\alpha(5)=1$.

Onto set of permutation we can define the multiplication operation.

Definition. For any two permutation α & β onto set $\{1,2,\dots,n\}$ we define the permutation $\alpha\circ\beta$ by following condition $\alpha\circ\beta(i) = \alpha(\beta(i))$, $i=1,2,\dots,n$ i.e. on the i-th place of permutation placed the number which in permutation α placed on the $\beta(i)$-th place.

For example,

$\alpha(3,1,2)$ и $\beta=(1,3,2)$. $\alpha\circ\beta(1)=\alpha(\beta(1))=\alpha(1)=3$; $\alpha\circ\beta(2)=\alpha(\beta(2))=\alpha(3)=2$

$\alpha\circ\beta(3)=\alpha(\beta(3))=\alpha(2)=1$, i.e. $\alpha\circ\beta=(3,2,1)$.

From the other hand, $\beta\circ\alpha(1)=\beta(3)=2$; $\beta\circ\alpha(2)=\beta(1)=1$; $\beta\circ\alpha(3)=\beta(2)=3$, т.е. $\beta\circ\alpha=(2,1,3)$.
So, $\alpha\circ\beta\neq\beta\circ\alpha$.

Actually $\alpha\circ\beta(i)=\beta\circ\alpha(i)$ if in the permutation α on the $\beta(i)$-th place stays the same number that stays on the $\alpha(i)$-th place in permutation β. That in general is not true. However, if in the example we take $\beta=(2,3,1)$, then $\alpha\circ\beta(1)=\alpha(\beta(1))=\alpha(2)=1$; $\alpha\circ\beta(2)=\alpha(3)=2$; i.e. if $\alpha=(3,1,2)$ & $\beta=(2,3,1)$, then $\alpha\circ\beta=\varepsilon=(1,2,3)$, and in this case $\beta\circ\alpha=\varepsilon$.

Let α,β,γ be a three arbitrary permutation, then $\alpha\circ(\beta\circ\gamma)=(\alpha\circ\beta)\circ\gamma$, i.e. the product of permutation is associative:

$\alpha\circ(\beta\circ\gamma)(i) = \alpha(\beta\circ\gamma(i)) = \alpha(\beta(\gamma(i)))$ & $(\alpha\circ\beta)\circ\gamma(i) = (\alpha\circ\beta)(\gamma(i)) = \alpha(\beta(\gamma(i)))$.

And clearly that $\alpha\circ\varepsilon=\varepsilon\circ\alpha$.

Let α be an arbitrary permutation, then there exist a permutation β such that $\alpha\circ\beta=\beta\circ\alpha=\varepsilon$. Actually, since $\{\alpha(1),\alpha(2),\dots,\alpha(n)\}=\{1,2,\dots,n\}$, for every $j\in\{1,2,\dots,n\}$ we can point i such that on the i-th place of permutation α placed j, i.e. $\alpha(i)=j$ and such i is unique for every j, since if $\alpha(i_1)=j$, then $\alpha(i)=\alpha(i_1)$, that's possible only if $i=i_1$. Assume $\beta(j)=i$, then $\alpha\circ\beta(j)=\alpha(i)=j$ for every j. Let $i\in\{1,2,\dots,n\}$ be chosen by arbitrary way then for $j=\alpha(i)$ by definition of β we have $\beta(j)=i$. Hence $\beta\circ\alpha(i)=i$. Therefore $\alpha\circ\beta=\beta\circ\alpha = \varepsilon$.
Permutation β is defined unique by condition $\alpha\circ\beta=\beta\circ\alpha=\varepsilon$, if β_1 satisfies condition $\beta_1\circ\alpha = \alpha\circ\beta_1 = \varepsilon$, then

$$\beta_1 = \beta_1\circ\varepsilon = \beta_1\circ(\alpha\circ\beta) = (\beta_1\circ\alpha)\circ\beta = \varepsilon\circ\beta = \beta.$$

Therefore such permutation has special notion α^{-1} and call inverse permutation to the permutation α.

Existence of inverse permutations α^{-1} for any permutation α makes it possible to solve equation in permutations. Let given two permutation α & β then equation $\alpha\circ\gamma=\beta$ relatively γ can be solved by following way:

$$\alpha^{-1}\circ\beta = \alpha^{-1}\circ(\alpha^{-1}\circ\gamma) = (\alpha^{-1}\circ\alpha)\circ\gamma = \varepsilon\circ\gamma = \gamma.$$

So, $\gamma = \alpha^{-1}\circ\beta$.

Similarly, equation $\gamma\circ\alpha = \beta$ has solution $\gamma = \beta\circ\alpha^{-1}$.
Now is time for examples. From example shown before we can see that for $\alpha=(3,1,2)$ $\alpha^{-1}=(2,1,3)$.

We're going to show how to build inverse permutations.

Let $\alpha=(4,1,3,5,2)$. $\alpha(1)=4 \Rightarrow 1=\alpha^{-1}(4)$;

$\alpha(2)=1 \Rightarrow 2=\alpha^{-1}(1)$; $\alpha(3)=3 \Rightarrow \alpha^{-1}=3$; $\alpha(4)=5 \Rightarrow \alpha^{-1}(5)=4$; $\alpha(5)=2 \Rightarrow \alpha^{-1}(5)=4$.

Hence, $\alpha^{-1}=(2,5,3,1,4)$.

Let $\beta=(3,2,5,1,4)$. Solve equation $\alpha\circ\gamma=\beta$, then $\gamma=\alpha^{-1}\circ\beta$.

$\gamma(1)=\alpha^{-1}(\beta(1))=\alpha^{-1}(3)=3$; $\gamma(2)=\alpha^{-1}(\beta(2))=\alpha^{-1}(2)=5$; $\gamma(3)=\alpha^{-1}(\beta(3))=\alpha^{-1}(5)=4$;

$\gamma(5)=\alpha^{-1}(\beta(5))=\alpha^{-1}(4)=1$. Hence, $\gamma=(3,5,4,2,1)$.

Definition. In what follows we shall say that two ordered sets $(a_1,a_2,\dots,a_n)$ & $(b_1,b_2,\dots,b_n)$ are concordant in order (or simply concordant), if all of ordered pairs (a_i,a_j) & (b_i,b_j), where $1\le i<j\le n$ are concordant in order.

Example. Sets $(7.5,3.4,-1,2.3)$ & $(5.1,2.5,0,1.4)$ are concordant.

Let us have ordered set $x=(x_1,x_2,\dots,x_n)$, then for every permutation α the set x correspond to ordered set

$$x_\alpha=(x_{\alpha(1)},x_{\alpha(2)},\dots,x_{\alpha(n)}),$$

which we call *permutation of set* x corresponded to the permutation α.

Remind you that for two ordered sets x & y $S(x,y)=x_1y_1+x_2y_2+\dots+x_ny_n$

Clearly, for any permutation α $S(x_\alpha,y_\alpha)=S(x,y)$.

Theorem 5. Let $x=(x_1,x_2,\dots,x_n)$ & $y=(y_1,y_2,\dots,y_n)$ be two ordered sets such that $x_1\le x_2\le\dots\le x_n$ & $y_1\le y_2\le\dots\le y_n$. Then for any permutation α onto set $\{1,2,\dots,n\}$: $S(x,y_\alpha)\le S(x,y)$, i.e.

$$x_1y_{\alpha(1)}+x_2y_{\alpha(2)}+\dots+x_ny_{\alpha(n)}\le x_1y_1+x_2y_2+\dots+x_ny_n \tag{37}$$

Proof: (*induction by* n)

1. *Base.* n = 2. It's equivalent to the definition of concordant pairs.
2. *Induction.* Let $n>2$. Suppose that the theorem is true for n-1

Let α be an arbitrary permutation onto set $\{1,2,\dots,n\}$. Here is possible two cases:

1. $\alpha(n)=n \Rightarrow x_1y_{\alpha(1)}+x_2y_{\alpha(2)}+\dots+x_ny_{\alpha(n)}=x_1y_{\alpha(1)}+\dots+x_{n-1}y_{\alpha(n-1)}+x_ny_n$

Then α define a permutation onto set $\{1,\dots,n\}$ by supposition of induction

$x_1y_{\alpha(1)}+x_2y_{\alpha(2)}+\dots+x_ny_{\alpha(n-1)}\le x_1y_1+x_2y_2+\dots+x_ny_{n-1}\Rightarrow S(x,y_\alpha)\le S(x,y)$.

2. $\alpha(n)\ne n$. Since there is a unique $j\in\{1,2,\dots,n\}$ such that $\alpha(j)=n$, i.e. $j=\alpha^{-1}(n)$, then $S(x,y_\alpha)=x_ny_{\alpha(n)}+x_jy_n+\overline{S}$, where $\overline{S}$ is remain terms in sum $S(x,y_\alpha)$. Pairs (x_n,y_j) & $(x_n,y_{\alpha(n)})$ are concordant since $n>j$ & $n>\alpha(n)$, and therefore $x_ny_{\alpha(n)}+x_jy_n\le x_ny_n+x_jy_{\alpha(n)}$.

It follows that

$S(x,y_\alpha)\le x_ny_n+x_jy_{\alpha(n)}+\overline{S}=S(x,y_\beta)$, where β is a permutation of $\{1,2,\dots,n\}$ defined by following way $\beta(n)=n$, $\beta(j)=\alpha(n)$, $\beta(i)=\alpha(i)$ for any $i\ne n$ & $i\ne j$.

Now use case 1 considered before to the permutation β and sum $S(x,y_\beta)$.

The theorem is proved.■

Corollary 1. When holds condition of theorem the following inequality is true

$$x_1y_n+x_2y_{n-1}+\dots+x_ny_1\le x_1y_{\alpha(1)}+x_2y_{\alpha(2)}+\dots+x_ny_{\alpha(n)}$$

where α is an arbitrary permutation onto set $\{1,2,\dots,n\}$.

Proof. Sets $(x_1, x_2, \ldots, x_n)$ & $(-y_1, -y_2, \ldots, -y_n)$ are concordant and satisfy the theorem condition and therefore

$$x_1(-y_{\alpha(1)}) + x_2(-y_{\alpha(2)}) + \ldots + x_n(-y_{\alpha(n)}) \le x_1(-y_n)+x_2(-y_2)+\ldots+x_n(-y_n) \Leftrightarrow$$

$$\Leftrightarrow x_1 y_{\alpha(1)} + x_2 y_{\alpha(2)} + \ldots + x_n y_{\alpha(n)} \ge x_1 y_n + x_2 y_{n-1} + \ldots + x_n y_1 .$$

Corollary 2. Let sets $(x_1, x_2, \ldots, x_n)$ & $y=(y_1, y_2, \ldots, y_n)$ be concordant, then for any permutation α onto set $\{1, \ldots, n\}$ holds inequality

$$x_1 y_{\alpha(1)} + x_2 y_{\alpha(2)} + \ldots + x_n y_{\alpha(n)} \le x_1 y_1 + \ldots + x_n y_n$$

Proof. Let β be a permutation of numbers $\{1, 2, \ldots, n\}$ such that $x_{\beta(1)} \le x_{\beta(2)} \le \ldots \le x_{\beta(n)}$. Then by concordance property $y_{\beta(1)} \le y_{\beta(2)} \le \ldots \le y_{\beta(n)}$ i.e. β is concordant permutation. The sets x_β & y_β satisfy theorem conditions.

Let α be an arbitrary permutation onto set $\{1, 2, \ldots, n\}$. Define permutation γ from correlation $\alpha\circ\beta=\beta\circ\gamma$, then $\gamma=\beta^{-1}\circ\alpha\circ\beta$.
Denote x_β & y_β by x' & y' respectively, i.e. $x'_i = x_{\beta(i)}$ & $y'_i = y_{\beta(i)}$, $i=1, \ldots, n$.
Then by theorem 5 $S(x', y'_\gamma) \le S(x', y')$.
But $S(x', y') = S(x_\beta, y_\beta) = S(x, y)$ & $S(x', y'_\gamma) = S(x_\beta, y_{\beta\circ\gamma})$,
since $y'_{\gamma(i)} = y_{\beta(\gamma(i))}$, $i=1, 2, \ldots, n$. As long as $\beta\circ\gamma=\alpha\circ\beta$, then
$S(x', y'_\gamma) = S(x_\beta, y_{\alpha\circ\beta}) = S(x, y_\alpha)$. Thus $S(x, y_\alpha) \le S(x, y)$.

Definition. Any two ordered sets $x=(x_1, x_2, \ldots, x_n)$ & $y=(y_1, y_2, \ldots, y_n)$ are to be said co-concordant if sets x & $-y$ are concordant.

Corollary 3. For co-concordant sets x & y and arbitrary permutation α onto set $\{1, 2, \ldots, n\}$ holds inequality:

$$S(x, y) \le S(x, y_\alpha)$$

Proof:

$$-S(x, y_\alpha) = S(x, -y_\alpha) \le S(x, -y) \Leftrightarrow S(x, y_\alpha) \ge S(x, y).$$

Sets x & y are co-concordant if and only if for any $1 \le i < j \le n$ $(x_i - x_j)\cdot(y_i - y_j) \le 0$. Really, x & y are co-concordant $\Leftrightarrow$ x & -y are concordant $\Leftrightarrow$ $(x_i - x_j)\cdot(-y_i - (-y_j)) \ge 0$ $\Leftrightarrow$ for any $1 \le i < j \le n$ $(x_i - x_j)\cdot(y_i - y_j) \le 0$.

Example. Sets (a, b, c) & (bc, ca, ab) are co-concordant:

$(a-b)\cdot(bc-ca) = -c(a-b)^2 \le 0$; $(b-c)\cdot(ca-ab) = -a(b-c)^2 \le 0$; $(a-c)\cdot(bc-ab) = -b(a-c)^2 \ge 0$

For positive $x_1, x_2, \dots, x_n$ sets $(x_1, x_2, \dots, x_n)$ & $\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)$ are co-concordant. Actually, $(x_i - x_j)\cdot\left(\frac{1}{x_i} - \frac{1}{x_j}\right) = -\frac{(x_i-x_j)^2}{x_i x_j} \le 0$.

Corollary 4. Let $x=(x_1, x_2, \dots, x_n)$ & $y=(y_1, y_2, \dots, y_n)$ are concordant sets then holds *Tchebysheff* inequality (see inequalities (34) & (35)).

$$n\cdot(x_1y_1+x_2y_2+\dots+x_ny_n) \ge (x_1+x_2+\dots+x_n)\cdot(y_1+y_2+\dots+y_n)$$

and if $x_1+x_2+\dots x_n>0$, then

$$\frac{x_1y_1+x_2y_2+\dots+x_ny_n}{x_1+x_2+\dots+x_n} \ge \frac{y_1+y_2+\dots+y_n}{n}$$

Proof:

$$S(x,y) \ge x_1y_1 + x_2y_2 + \dots + x_{n-1}y_{n-1} + x_ny_n$$

$$S(x,y) \ge x_1y_2 + x_2y_3 + \dots + x_{n-1}y_n + x_ny_1$$

$$S(x,y) \ge x_1y_3 + x_2y_4 + \dots + x_{n-1}y_1 + x_ny_2$$

....

$$S(x,y) \ge x_1y_n + x_2y_1 + \dots + x_{n-1}y_{n-2} + x_ny_{n-1}$$

By addition those inequalities we get

$$n\cdot S(x,y) \ge x_1\cdot(y_1+y_2+\dots+y_n) + x_2\cdot(y_1+y_2+\dots+y_n) + \dots + x_n\cdot(y_1+y_2+\dots+y_n)$$

$$\Leftrightarrow \quad n\cdot(x_1y_1+x_2y_2+\dots+x_ny_n) \ge (x_1+\dots+x_n)\cdot(y_1+\dots+y_n).$$

Similarly, we get $n\cdot(x_1y_n+x_2y_{n-1}+\dots+x_ny_1) \le (x_1+\dots+x_n)\cdot(y_1+\dots+y_n)$.

Exercise 41. Denote the cyclic permutation on 1-st element by α.
α defines by following way: $\alpha(1)=2$; $\alpha(2)=3$; ...; $\alpha(n-1)=n$; $\alpha(n)=1$, i.e.

$\alpha(i)=\begin{cases} i+1, & \text{если } 1\le i\le n-1 \\ 1, & \text{если } i=n \end{cases}$. And let $\alpha^k=\alpha\circ\alpha\circ\dots\circ\alpha$ (k factors).

Proof:

1. $\alpha(i) = n\cdot\{\frac{i+1}{n}\}$, where $\{x\}$ is fractional part of x.
2. $\alpha^k(i) = n\cdot\{\frac{i+k}{n}\}$, for any k.
3. $\alpha^n=\varepsilon$.
4. $x_{\alpha^k(1)} + x_{\alpha^k(2)} + \dots + x_{\alpha^k(n)} = x_1+x_2+\dots+x_n$
5. Tchebysheff inequality by using cyclic permutations.

By using concordant sets we can give one more proof of Cauchy's inequality (8)

Denote $\sqrt[n]{a_1a_2\ldots a_n}$ by b. Easy to notice that

sets $\left(\frac{a_1}{b}, \frac{a_1a_2}{b^2}, \ldots, \frac{a_1a_2\ldots a_n}{b^n}\right)$ & $\left(\frac{b}{a_1}, \frac{b^2}{a_1a_2}, \ldots, \frac{b^n}{a_1a_2\ldots a_n}\right)$ are co-concordant

Hence,

$$\frac{a_1}{b}\cdot\frac{b}{a_1} + \frac{a_1a_2}{b^2}\cdot\frac{b^2}{a_1a_2} + \ldots + \frac{a_1a_2\ldots a_n}{b^n}\cdot\frac{b^n}{a_1a_2\ldots a_n} \le \frac{b}{a_1}\cdot\frac{a_1a_2}{b^2} + \frac{b^2}{a_1a_2}\cdot\frac{a_1a_2a_3}{b^3}$$

$$+ \ldots + \frac{b^{n-1}}{a_1a_2\ldots a_{n-1}}\cdot\frac{a_1a_2\ldots a_n}{b^n} + \frac{b^n}{a_1a_2\ldots a_n}\cdot\frac{a_1}{b} \iff n \le \frac{a_2}{b}+\frac{a_3}{b} + \ldots + \frac{a_n}{b}+$$

$$+ \frac{a_1a_2\ldots a_n}{a_1a_2\ldots a_n}\cdot\frac{a_1}{b} = \frac{a_1+a_2+\ldots+a_n}{b} \iff \frac{a_1+a_2+\ldots+a_n}{n} \ge b = \sqrt[n]{a_1a_2\ldots a_n}.$$

Exercise 42. Prove inequalities

a) $3\cdot(a^3+b^3+c^3) \ge (a+b+c)\cdot(a^2+b^2+c^2)$, $a,b,c\ge 0$

b) $(a_1+a_2+\ldots+a_n)\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) < n\cdot\left(\frac{a_1}{a_n}+\frac{a_2}{a_{n-1}}+ \ldots +\frac{a_n}{a_1}\right).$

Exercise 43. Let a, b, c be a sides of triangle and α,β,γ are its angles lying opposite the sides a, b, c respectively. Prove inequalities

a) $\alpha\cdot\cos\alpha + \beta\cdot\cos\beta + \gamma\cdot\cos\gamma \le \frac{\pi}{2}$

b) $\frac{a\cdot\cos\alpha + b\cdot\cos\beta + c\cdot\cos\gamma}{a+b+c} \le \frac{1}{2}$

c) $\frac{\pi}{3} \le \frac{a\cdot\alpha + b\cdot\beta + c\cdot\gamma}{a+b+c}$

d) $\frac{\sqrt{3}}{2} \le \frac{a\cdot tg\frac{\alpha}{2} + b\cdot tg\frac{\beta}{2} + c\cdot tg\frac{\gamma}{2}}{a+b+c}$

e) let $0<\alpha_1<\alpha_2<\ldots<\alpha_n<\frac{\pi}{2}$, $\frac{\sin\alpha_1+\sin\alpha_2+\ldots+\sin\alpha_n}{\cos\alpha_1+\cos\alpha_2+\ldots+\cos\alpha_n} < \frac{1}{n}(tg\,\alpha_1+\ldots+tg\,\alpha_n)$

*(Hint. In all inequalities of exercises **42,43** use Tchebysheff inequality. In exercise **43** (a,**b**) prove* $\frac{\cos\alpha+\cos\beta}{2} \le \cos\frac{\alpha+\beta}{2}$ *for* $0<\alpha,\ \beta<\frac{\pi}{2}$, *and* $\cos\frac{\alpha+\beta+\gamma}{3} \ge \frac{\cos\alpha+\cos\beta+\cos\gamma}{3}$ *by the scheme of doubling + reverse step).*

<u>*To be continued.*</u>

FOR YOUNG PUPILS...

Problems

1. Guessing how was constracting first table insert the insufficient number. Do the same with second table and remove exceed number.

5	625	4
8	8	1
7	?	2
6	216	3

table 1

$\frac{1}{3}$	$\frac{1}{8}$	$\frac{23}{7}$
$3\frac{2}{7}$	$\frac{4}{11}$	$0.(3)$
0.12	$\frac{5}{13}$	$0.(36)$

table 2

2. What is three-valued number equal to the cube of latest figure of the number and simultaneously equal to the square of number formed by first & second figure ?

3. One of the friends said to another that made up an exercise on division in which divident, divisor, quotient and remainder finishing on 1,3,5,7 respectively. Does it possible ?

4. In triangle ABC taken an arbitrary point . Prove that MB + MC < AB + AC. (*Fig.* 1)

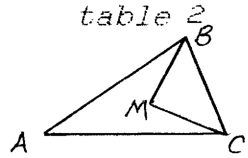


Fig. 1

5. In an exercise on multiplication (*Fig.* 2) every star mean the simple one-valued number (2,3,5 or 7). Restore the numbers.

```
     ***
  ×  ***
    ****
   ****
  ****
  ******
```

Fig. 2

6. On hypotenuse of right triangle outside constructed a equilateral triangle with the area in twice more than area of right triangle. Find its acute angles.

7. Moshe and Yosi want to buy a bubble. Moshe need another 15 agorot and Yosi need 1 agorot. When they add the money it wasn't enough again. How much does bubble coast ?

8. Write into cells in figure 3 all numbers 1 up to 10 so that showed equalities hold true.

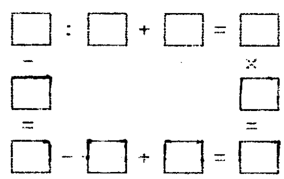


Fig. 3

9. Solve in integer numbers equation
$$2xy = 5x + 3y.$$

10. Given that some number by division by 1991 and 1992 gives the same remainder 924. What's the remainder gave this number by division by 543 ?

(Problems has been collected by Alt Arkady).

"DELTA"'S COMPETITION

1. Prove that there are infinite number of pairs of natural numbers x,y such that

$\frac{x^2+p}{y}$ & $\frac{y^2+p}{x}$ are integer, where p is a natural number.

2. Prove that derivative of function

a) $f(x) = \frac{x-1}{x-2}\cdot\frac{x-2}{x-3}\cdot\frac{x-3}{x-4}\cdot\ldots\cdot\frac{x-2n+1}{x-2n}$

b) $f(x) = \frac{(x-a_1)\cdot(x-a_3)\cdot\ldots\cdot(x-a_{2n+1})}{(x-a_2)\cdot(x-a_4)\cdot\ldots\cdot(x-a_{2n})}$ with $a_1<a_2<\ldots<a_{2n+1}$

is negative for all points from the domain of definition of f(x).

3. Prove that if a,b,c,d are natural numbers and $a\cdot b=c\cdot d$, then $a^{1992}+b^{1992}+c^{1992}+d^{1992}$ is composite number

4. Solve system of equation

$$\begin{cases}(b+c)\cdot(x+\frac{1}{x}) = (a+c)\cdot(y+\frac{1}{y}) = (a+b)\cdot(z+\frac{1}{z})\\ xy+yz+xz=1.\end{cases}\quad, a,b,c>0.$$

5. Given sequence

$\begin{cases}x_1=\frac{1}{3}\\ x_{n+1}=x_n^3+x_n, n=1,2,..\end{cases}$. Find greatest integer of $\frac{x_1}{x_1^2+1}+\frac{x_2}{x_2^2+1}+\ldots+\frac{x_{1991}}{x_{1991}^2+1}$

6. Function f(x) defined onto segment [0,1] and satisfy equation $f(x+f(x)) = f(x)$, where $x\in[0,1]$. Prove that $f(x)=0$ for all $x\in[0,1]$.

7. Solve system of equation

$$\begin{cases}x-y=\sin x\\ y-z=\sin y\\ z-x=\sin z\end{cases}$$

8. Prove that the sum of squares of distances from arbitrary point lying on the circle to the equilateral triangle inscribed into circle is a constant value. Find it.

9. Prove inequality

$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \le \frac{\sqrt{3(a^2+b^2+c^2)}}{4\cdot S}$, where m_a, m_b, m_c are medians & h_a, h_b, h_c are altitudes dropped to side a,b,c respectively, S is area of triangle.

10.* Prove that for all $n\in\mathbb{N}$, $n\ge 2$ the greatest value of function $F(x_1,\ldots,x_n)=(x_1+\ldots+x_n)\cdot(\frac{1}{x_1}+\ldots+\frac{1}{x_n})$ onto segment [a,b] is $n^2+[\frac{n^2}{4}]\cdot\frac{b-a}{ab}$

(Problems have been prepared by Alt Arkady, "Blikh" school, Ramat-Gan).

QUICK PROBLEMS

Here we offer to the reader unordinary problems and their solution could be demonstrated within a minute. And understandable solution can be written on a line. However, such problem are psychologically difficult problem and their solution distinct by brevity only if you got an one unexpected idea (!) of solution, and frequently difficult to find such idea. This searching can take a lot of time at least not a minute, although some of you maybe can do it within a minute. Solution of those problem we show in department "Answers. Hint. Solutions." giving to the reader some time to think.

1. In triangle two altitudes no less than length of sides to which them were dropped. Find angles of the triangle.

2. Prove that for any natural n>1 number $3^{2n+1} - 2^{2n+1} - 6^{2n}$ is composite.

3. For three polynomial P, Q, G find polynomial F such that F greater than each of given polynomials (i.e. for all x F(x) has value greater than values of P, Q, G).

MONSTER PROBLEMS

Here we offer to the reader a series of problem possessing one peculiarity – they look very strange. Moreover, they shock. From the first vision on those problem you can say that they are something impossible and connection of unconnectable things. You could say that authors in purpose makes them so complicate. And you'd be right, but not completely because the reader whose be able to overcome this shock reaction can understand that this shock factor is a kind of hint. After that is matter of technic and good background of traditional school base. The aim of those problem is to learn how to breakthru such psychologic barriers, to test your knowledge on level of searching and combinations necessary tools. The solution of the first one you can see in the department "Answers. Hint. Solutions" and we let you thing about another. We demonstrate the solution of the first problem to whom gave up and solution of second problem wait for you in the next issue.

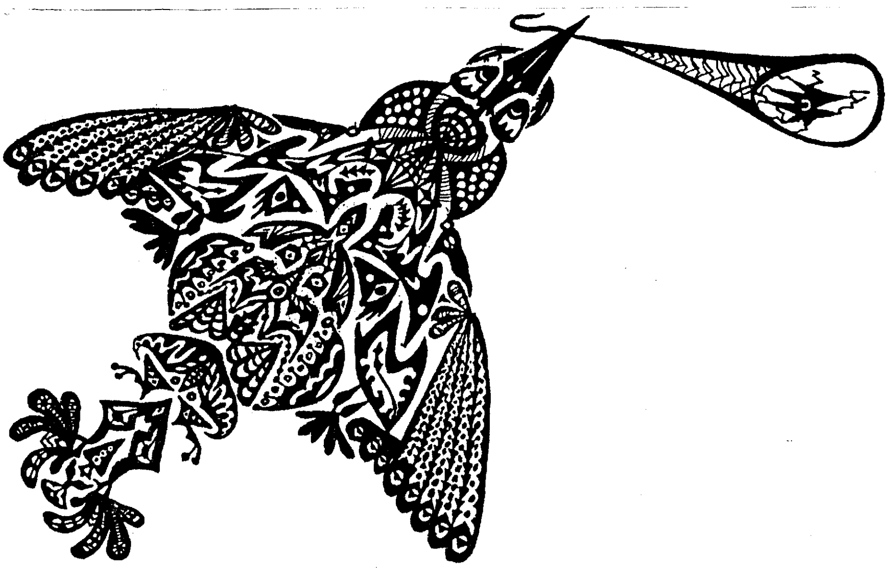
1. Find all pairs (x,y) satify system

$$\begin{cases} y\cdot\sin x = \log_2\left|\dfrac{y\cdot\sin x}{1+3y}\right| \\ (6y^2+2y)\cdot\left(4^{\sin^2 x}+4^{\cos^2 x}\right) = 25\cdot y^2+6y+1 \\ |y|\le 1 \end{cases}$$

2. Find all values of parameter a when equation

$$\log_{1/\pi}\left(\frac{a^2+4\pi^2+4}{4\cdot x-x^2-2\cdot(a-2\pi)\cdot|x-2|+4\pi\cdot a}\right)-\sqrt{(x-5a+10\pi-34)\cdot|\pi-x|-a+\pi+2}=0$$

has at least one integer solution.



OLYMPIADS

31-TH INTERNATIONAL MATHEMATICAL COMPETITION

Current 31-th international mathematical olympiad took place in Pekin from 7-th to 19-th of july, 1990. In the olympiad took part 308 pupils from 54 countries. First time in the olympiad took part teams from Algire, Bachrein, Japan, Macao and North Korea. The olympiad occur in two rounds (3 problems for each round, for their solution was given 4.5 hours for each round). Solution of every problem was scored by 7 points. Pupils have been rewarded by gold medal if their score was 34-42 points, by silver medal if their score was 23-33 points and by bronze medal if their score was 16-22 points. In following table you can see the top results of ten best team of this year.

China	USSR	USA	Romania	France	Hungary	Germany	Bulgary	U.K.	Chekhoslov.
42	42	39	36	42	34	33	38	40	33
42	40	35	34	36	33	30	29	39	30
41	40	33	32	34	29	27	23	19	26
36	27	32	29	28	24	25	23	18	24
36	25	23	22	15	22	22	23	13	24
33	19	12	18	13	20	21	16	12	16
230	193	174	171	168	162	158	152	141	153

Now, look at problems (in brackets you see the country which was offered the problem). Solutions see in next issue.

1. *(India). Chords* AB *and* CD *intersect at point* E *into given circle. Let* M *be a point of segment* BE. *The tangent touched the circle at point* E *and which going through points* D,E & M *it intersect the lines* BC & AC *at points* F & G *respectively. Let* AM/AB = t. *Find* EG/EF *as a function of* t.

2. *(Chekhoslovakya). On circle given the set* E *of* (2n-1) *distinct points* ($n \geq 3$). k *of them painted in black colour and the rest of them painted in white colour. The painting is to be said good if there are two black points between which the arc contain exactly* n *points from the set* E. *Find a smallest value of* k *for which every painting of point from the set* E *is good.*

3. *(Romania). Find all integer* $n>1$ *such that* $\dfrac{2^n+1}{n^2}$ *is integer.*

4. *(Turkey). Let* Q^+ *is set of all rational positive numbers. Find an example for function* $f: Q^+ \longrightarrow Q^+$ *such that*

$$f(x \cdot f(y)) = \frac{f(x)}{y} \quad \text{for all } x,y \in Q^+.$$

5. (Germany). Given a natural number $n_0 > 1$. *Players* A & B *choose one after another natural numbers* $n_1, n_2, \ldots$ *by following inductive rule. Player* A *knew the number* n_{2k} *can choose any number* n_{2k+1} *such that*

$$n_{2k} \le n_{2k+1} \le n_{2k}^2$$

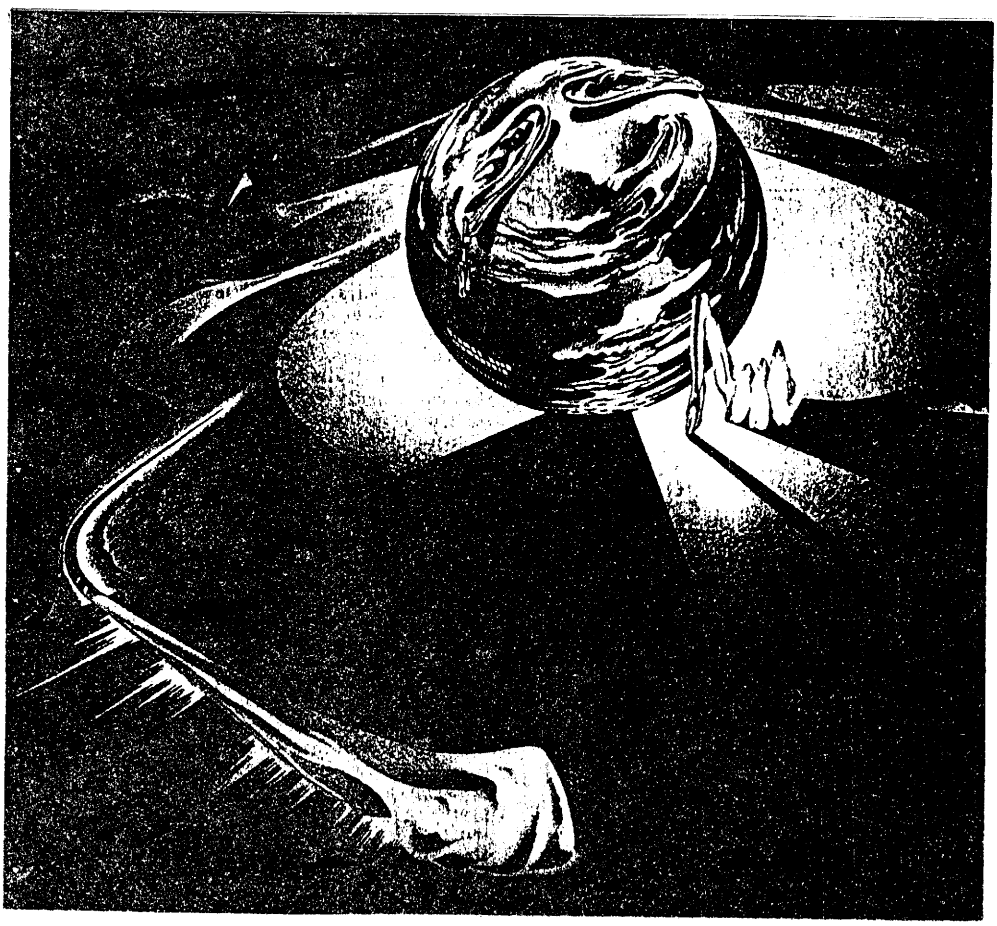
After that player B *chooses any number* n_{2k+2} *such that* n_{2k+1}/n_{2k+2} *is a natural degree of a prime number. Player* A *win if he choose* 1990 *and player* B *win if he chooses* 1.

Find all values of n_0 *for which*

a) A *has a winning strategy;*
b) B *has a winning strategy;*
c) *neither* A, B *have a winning strategy.*

6. (Holland). Prove that there exist a convex polygon with 1990 *sides such that*

a) *all its angles are equal;*
b) *lengths of polygon sides are equal to numbers* $1^2, 2^2, \ldots, 1989^2, 1990^2$ *in some order.*



ANSWERS. HINTS. SOLUTIONS.

Problems for young pupils.

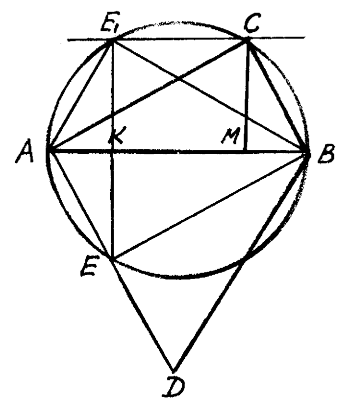
1. In the first table you should write to insert $7^2=49$, since the rule of writting cells is | x | x^k | k |.

In the second table the exceed number is 5/13, since other cells are filled by pairs of equal numbers: $\frac{1}{3}=0.(3)$; $3\frac{2}{7}=\frac{23}{7}$; $\frac{1}{8}=0.125$; $\frac{4}{11}=0.(36)$.

2. Answer. $729 = 9^3 = 27^2$.

3. No, impossible. Since if a – divident, b – divisor, k – quotient, r – remainder, then $a=k\cdot b+r$ and last figure of number is a last figure of the result by corresponded operation over lasts figures k, b & r, i.e. the last figure of nuber is $3\cdot 5+7=22$.

4. Draw BM up to intersection with the side AC at point E. Then: $AE+AB > EB \Rightarrow AE+EC+AB > EB+$ $+EC = EC + (EM+MB) = (EC+EM) + MB > MC+MB$.



5. Answer. $325\times 777=252525$.

6. Draw from the vertex B of equilateral triangle a altitude BE to side AD. Then the area of triangle AEB is equal to the area of triangle ABC, since altitude BE is the median of triangle ADB too. Therefore, the area of triangle AEB is equal to half area of triangle ADB. Hence, altitude EK of right triangle AEB is equal to altitude CM of triangle ABC. Thus, triangles AEB and BCA are equal. Actually, into half-circle with diameter AB we can construct exactly two equal right triangles of given altitude. (It's possible a algebraic way to prove the equality of right triangles by hypothenuse and altitude). From the equality of triangles we deduce that in ABC acute angles 30° & 60°.

7. Denote by x the price of bubble. Then Moshe has $0\le x-15$ agorot and Yosi has $0\le x-1$ agorot. By problem's condition: $x-15 + x-1 < x \iff x<16$. But $x\ge 15$. Thus, $x=15$ agorot. Hence, Yosi had 14 agorot, and Moshe had no money at all.

8. Answer.

$$\begin{array}{ccccccc} \boxed{8} & : & \boxed{4} & + & \boxed{3} & = & \boxed{5} \\ - & & & & & & \times \\ \boxed{1} & & & & & & \boxed{2} \\ = & & & & & & = \\ \boxed{7} & - & \boxed{6} & + & \boxed{9} & = & \boxed{10} \end{array}$$

Quick problems.

1. $\begin{cases} h_a \ge a \\ h_b \ge b \end{cases} \Rightarrow h_a \ge a \ge h_b \ge b \ge h_a \Rightarrow$

$\Rightarrow h_a=a=b=h_b$. Triangle is issosceles and right. Angles $90^\circ, 45^\circ, 45^\circ$.

2. $3^{2n+1} - 2^{2n+1} - 6^n = (3^{n+1}+2^{n+1})\cdot(3^n-2^n)$.

3. $F = P^2 + Q^2 + G^2 + \frac{1}{4}$

Monster problems.

1. Solve the second equation of the system as quadratic relatively $t=4^{\sin^2 x}$. Notice that $y\neq 0$ (otherwise $\log_2\left|\frac{y\cdot\sin x}{1+3y}\right|$ is undetermined), we get for the second equation in taken notion:

$$(6y^2+2y)\cdot t^2 - (25y^2+6y+1)\cdot t + 4\cdot(6y^2+2y) = 0 \Leftrightarrow t^2 - \left(\frac{3y+1}{2y} + \frac{8y}{3y+1}\right)\cdot t + 4=0$$

By Wiette's theorem:

$$\left[\begin{array}{l} t=\frac{3y+1}{2y} \\ t=\frac{8y}{3y+1} \end{array}\right.$$ Hence 2-nd equation becomes $$\left[\begin{array}{l} 4^{\sin^2 x} = \frac{3y+1}{2y} \\ 4^{\sin^2 x} = \frac{8y}{3y+1} \end{array}\right.$$

Consider the first case $4^{\sin^2 x} = \frac{3y+1}{2y}$. Since given that $|y|\le 1$ and $0\le\sin^2 x\le 1$, then y has to satisfy system of inequalities.

$$\left\{\begin{array}{l} y\neq 0 \\ |y|\le 1 \\ 1 \le \frac{3y+1}{2y} \le 4 \end{array}\right.$$. By straightforvard logarithm $4^{\sin^2 x} = \frac{3y+1}{2y}$ by the base 2, yields $2\cdot\sin^2 x + 1 = -\log_2\frac{y}{3y+1}$. By using this we can rewrite equality

е в виде: $2\cdot\sin^2 x + y\cdot\sin x + 1 = \log_2|\sin x|$.

But $2\cdot\sin^2 x + y\cdot\sin x + 1 = 2\cdot(\sin^2 x + 2\cdot\sin x\cdot\frac{y}{4} + \frac{y^2}{16}) + 1 - \frac{y^2}{8} =$
$= 2\cdot(\sin x + \frac{y}{4})^2 + \frac{8-y^2}{8} > 0$ for any $|y|\le 1$. It follows that

$$\log_2|\sin x| > 0 \Leftrightarrow |\sin x| > 1, \text{ that's impossible.}$$

Thus, given system equivalents

$$\left\{\begin{array}{l} y\cdot\sin x = \log_2\left|\frac{y\cdot\sin x}{1+3y}\right| \\ 4^{\sin^2 x} = \frac{8y}{3y+1} \\ |y|\le 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} y\cdot\sin x = \log_2|\sin x| + \log_2\frac{y}{1+3y} \\ 2\cdot\sin^2 x = \log_2\frac{y}{3y+1} + 3 \\ 1\le\frac{8y}{3y+1}\le 4 \\ |y|\le 1 \end{array}\right. \Leftrightarrow$$

$$\Rightarrow \left\{\begin{array}{l} -2\cdot\sin^2 x + y\cdot\sin y - 3 = \log_2|\sin x| \\ |y| \le 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} -2\cdot\sin^2 x + y\cdot\sin x + 3 \le 0 \\ |y| \le 1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} 2\cdot\sin^2 x - y\cdot\sin x - 3 \ge 0 \\ |y| \le 1 \end{array}\right.$$. But $\max\limits_{|y|\le 1}(2\cdot\sin^2 x - y\cdot\sin x - 3) =$

$= \max\{2\cdot\sin^2 x - \sin x - 3,\ 2\cdot\sin^2 x + \sin x - 3\} \le 0$, i.e. for any x,y such

$= \max\{2\cdot\sin^2 x - \sin x - 3,\ 2\cdot\sin^2 x + \sin x - 3\} \le 0$, i.e. for any x,y such that $|y| \le 1$ $2\cdot\sin^2 x - y\cdot\sin x - 3 \le 0$ and the equality can be reached only if

$$\begin{cases} y = 1 \\ \sin x = -1 \end{cases} \quad \text{or} \quad \begin{cases} y = -1 \\ \sin x = 1 \end{cases}$$

Thus, as equivalent from the given system

$$\left[\begin{array}{l} \begin{cases} y = 1 \\ \sin x = -1 \end{cases} \\ \begin{cases} y = -1 \\ \sin x = 1 \end{cases} \end{array} \right.$$

. And the first pair $\begin{cases} y = 1 \\ \sin x = -1 \end{cases}$ doesn´t satisfy to the system, since $4^{\sin^2 x} = 4$ when $\sin x = -1$ and $\frac{8y}{3y+2}$ when $y = 1$.

So, we have that pair $\begin{cases} y = -1 \\ \sin x = 1 \end{cases}$ satisfies the given system.

Answer: $y = -1$, $x = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$.

RELAXATION CORNER

Militaty secrets:

- In military situation the value of sine can reach "4"
- The military secret is not what you're learning, the secret is that you are learning this.

On exams:

- Do you hope that your grade on exam will be "60" ? Yes,.you get it, but it doesn't help you to grow easy.
- What do you, mr. student, draw such unequal square ? Are you daltonic ?

Observation:

- The face on the foto must be quadratic

On lecture:

- You always have to remember that anything you do you do incorrect.
- Students ! Pi is the irrational number, otherwise it is equal to 3.14.
- This integral so simple that can be taken without "dx".
- The essence of gravitation power I prove within a couple of minutes.

Advice:

- You need to clean a car by hot water since increasing the stickness of atoms.

On the lecture on civil defence:

- During nuclear attack is necessary to entrench oneself. Shown by statistic data that a corpses of unentrenched soldiers have huge number of burns. Just as a corpses of entrenched soldiers have no burns.
- The zone of hitting is consist of zones A,B,C,D. The epicentre situates at the zone A. And besides very interesting that the bomb is falling just at the epicentre.

Definitions:

- What's a lattice ? The lattice is a metal sheet with broken into it holes.
- Ellips is a circle inscribed into square 3x4
- Bush is a totality of branches and leaves sticking out from an one place.